# EVERY HOMOTOPY THEORY OF SIMPLICIAL ALGEBRAS ADMITS A PROPER MODEL

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Abstract. We show that any closed model category of simplicial algebras over an algebraic theory is Quillen equivalent to a proper closed model category. By "simplicial algebra" we mean any category of algebras over a simplicial algebraic theory, which is allowed to be multi-sorted. The results have applications to the construction of localization model category structures.

## 1. Introduction

To axiomatize the notion of a "homotopy theory" Quillen introduced closed model categories [Qui67], and produced a number of examples of such, one class of which are categories of *simplicial algebras*. A standard technique for constructing new model categories from old ones is that of *localization*: given a category **C** equipped with a model category structure and a morphism $f$ in that category, one produces a new model category structure on **C** in which the weak equivalences are the smallest class containing both the old weak equivalences and the map $f$. There are several "machines" for constructing localization model category structures; one of the most general is due to Hirschhorn [Hir]; note also [DHK], [Smi], and [GJ99, Ch. X]. They have been used extensively in recent years, notably to construct model categories for stable homotopy theories.

These localization machines require that the initial model category structure on **C** have certain additional properties, beyond those introduced by Quillen. In most cases they require in particular that **C** be a "left proper" model category; namely, the class of weak equivalences should be closed under cobase change along cofibrations (see (2.1)). Properness was first introduced by Bousfield and Friedlander [BF78] as an axiom needed to be able to put a model structure on a category of spectra; their construction is in fact an instance of a localization model category.

Many well-understood examples of model categories, including most categories of simplicial algebras, turn out *not* to be proper. (We give examples of such in §2.10.) It is the most non-trivial axiom needed for the localization machines. Thus, the following question becomes significant: does our homotopy theory admit a *proper* model? That is, given a closed model category **C** which is not necessarily proper, does there exist a proper closed model category $\mathbf{C}'$ which has the same homotopy theory as **C**?

In this paper, we examine the case of *simplicial algebras*, i.e., simplicial objects in a category of algebras associated to an algebraic theory in the sense of Lawvere [Law63], and more generally the case of simplicial algebras over a *multi-sorted, simplicial theory* (see §4.1, §4.11). This class of examples includes simplicial groups, rings, and so forth, as well as algebras over a simplicial operad, as in [Rez96]. They are the simplicial analogues of the topological theories considered by Boardman and Vogt [BV73]. Categories of simplicial algebras always always admit a model category structure, with the weak equivalences being those of the underlying simplicial sets (7.2).

**Theorem A.** *The homotopy theory of a category of simplicial algebras always admits a proper model.*

*Date*: March 10, 2000.
1991 *Mathematics Subject Classification.* Primary 18G55; Secondary 18C10, 55U35.
The author was supported by an AMS Centennial Fellowship.

Whether *any* reasonable homotopy theory (e.g., one associated to a model category) admits a proper model is an open question; Theorem A is the only result in this direction that I am aware of.

Theorem A can be made more precise. It is a corollary of the following

**Theorem B.** *Let $T$ be a (possibly simplicial, possibly multi-sorted) theory, and let $T$-alg be the corresponding category of simplicial $T$-algebras, equipped with a simplicial model category structure in which a map is a weak equivalence or fibration if it is a weak equivalence or fibration of the underlying simplicial sets.*

*Then there exists a morphism $S \to T$ of simplicial theories such that*

(1) *the induced adjoint pair $S$-alg $\rightleftarrows T$-alg is a Quillen equivalence of model categories, and*
(2) *$S$-alg is a* proper *simplicial closed model category.*

The proof of Theorem B follows a straightforward pattern; we (a) put a model category structure on the category of simplicial theories (7.2) so that in particular cofibrant resolutions of simplicial theories exist; (b) show that algebras over a *cofibrant* simplicial theory are a proper model category (11.4); and (c) observe that weakly equivalent simplicial theories give rise to Quillen equivalent model categories of algebras (8.6).

We say that a category is *pointed* if the initial object is isomorphic to the terminal object. It is most natural to study stable homotopy of algebras in the context of pointed objects. Thus we offer

**Theorem C.** *Given the hypotheses of Theorem B, suppose that in addition $T$-alg is a pointed category. Then $S$ can be chosen as in Theorem B so that $S$-alg is also a pointed category.*

Finally, we have

**Theorem D.** *Given the hypotheses of Theorem B (resp. of Theorem C), the theory $S$ can be chosen as in Theorem B (or Theorem C) so that $S$-alg is a* cellular model category *in the sense of Hirschhorn* [Hir] .

By Hirschhorn's results [Hir], Theorem D implies

**Corollary.** *For any set of maps in $S$-alg there is a localization model category structure with respect to this set.*

The proofs of Theorems A, B, C and D are given in §12.

In order to prove these results, we need to set up a certain amount of foundations for algebraic theories and their homotopy theory; this will take all of §§3–8. Our exposition of theories (§§3–4) is more involved than one might like; this is because we want to deal with "multi-sorted" theories, and because we need to introduce the notion of "bimodules" of algebraic theories. However, this is not idle generalization: the category of single-sorted theories and categories of bimodules over such are themselves categories of algebras over a multi-sorted theory, so considering multi-sorted theories from the start lets us avoid much duplication of exposition. The theory of bimodules of algebraic theories plays an important role in the proofs of the main theorems (see §8 and §11).

Some of this foundational material seems to be of independent interest, notably our definition of "bimodules" of algebraic theories and their relation to functors between categories of algebras (4.4), and the homotopy invariance results (8.5) and (8.6).

1.1. **Notation and conventions.** We write $X\backslash\mathbf{C}$ and $\mathbf{C}/X$ for the categories of objects under and over a given object $X$ (the "comma categories"). We write $\mathbf{D}^{\mathbf{C}}$ or $\mathrm{Func}(\mathbf{C},\mathbf{D})$ for the category of functors from $\mathbf{C}$ to $\mathbf{D}$.

If $X$ and $Y$ are algebras over some monad $T$, we adopt the convention of writing $X \coprod^{T\text{-alg}} Y$ or $X \amalg^T Y$ for the coproduct of $X$ and $Y$ in the category of $T$-algebras. An undecorated coproduct symbol means one taken in some underlying category, which typically is sets or simplicial sets.

We write $\mathcal{S}$ for the category of sets, and $s\mathcal{S}$ for the category of simplicial sets; it is often convienient to regard $\mathcal{S} \subset s\mathcal{S}$ as the full subcategory of *discrete* simplical sets. Generally, we write $s\mathbf{C}$ for the category of simplicial objects in $\mathbf{C}$. The diagonal functor $\mathrm{diag}\colon s(s\mathbf{C}) \to s\mathbf{C}$ sends $\{Y_{p,q}\} \mapsto \{Y_{n,n}\}$. We often use the diagonal principle [GJ99, IV.1.7], which says that if $f\colon X \to Y$ is a morphism in $s(s\mathcal{S})$ (i.e., of bisimplicial sets) such that $f_{p,*}\colon X_{p,*} \to Y_{p,*}$ is a weak equivalence of simplicial sets for every $p \geq 0$, then $\mathrm{diag}(f)$ is a weak equivalence of simplicial sets.

1.2. **Acknowledgments.** The author would like to thank Paul Goerss for conversations which improved the paper. The author would also like to thank Haynes Miller for suggesting an improved title.

1.3. **Organization of the paper.** In §2 we describe the notion of proper model categories and prove some key properties; we also give several examples of categories of simplicial algebras which are not proper. In §§3 and 4 we establish what we need for algebraic theories and their algebras over sets and simplicial sets. In the approach we take, algebraic theories are simply monads over sets (or graded sets) which commute with filtered colimits. We also establish the notion of a bimodule between theories, and identify them with a certain class of functors between categories of algebras. In §§5 and 6 we carry out some preparations needed for §7, in which we describe the model category structure on categories of simplicial algebras, and for §8, in which we show that the homotopy theory of algebras over a theory is a weak homotopy invariant of the theory, and that cofibrant right modules over a theory preserve all weak equivalences. In §9 we establish a criterion for a category of simplicial algebras to be proper, by generalizing an argument of Dwyer and Kan [DK80]. In §10 we give a description of free theories using trees, which is then used in §11 to show that a cofibrant theory gives rise to a proper model category of algebras. We give proofs of Theorems A–D in §12.

## 2. Proper model categories

By **model category**, we mean a closed model category in the sense of Quillen [Qui67], [Qui69]. (See also [Hov99], who defines model categories with a slightly stronger set of axioms than Quillen. However, everything in this section holds under Quillen's axioms.) We write $\mathrm{Ho}\,\mathbf{M}$ for the category obtained by formally inverting the weak equivalences in a model category $\mathbf{M}$.

2.1. **Definition of properness.** We recall the notion of a proper model category.

*Definition* 2.2. A model category $\mathbf{M}$ is **left proper** if for each pushout square in $\mathbf{M}$ of the form

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

in which $i$ is a cofibration and $f$ is weak equivalence, the map $g$ is weak equivalence.

Similarly, $\mathbf{M}$ is **right proper** if it satisfies the dual property involving pullback squares, fibrations, and weak equivalences.

A model category $\mathbf{M}$ is **proper** if it is both left proper and right proper.

2.3. **Under- and over-categories and properness.** Properness is most naturally understood as a statement about "families" of model categories which are parameterized by the objects of a fixed model category.

We say that a pair of adjoint functors $L\colon \mathbf{M} \rightleftarrows \mathbf{N} : R$ between model categories is a **Quillen pair** if the left adjoint $L$ takes cofibrations to cofibrations and the right adjoint $R$ takes fibrations to fibrations. The pair forms a **Quillen equivalence** if, in addition, for each cofibrant object $X$ in

$\mathbf{M}$ and fibrant object $Y$ in $\mathbf{N}$, a map $LX \to Y \in \mathbf{N}$ is a weak equivalence if and only if its adjoint $X \to RY \in \mathbf{M}$ is.

**Proposition 2.4.** *A Quillen pair as above gives rise to a derived adjoint pair* $\operatorname{Ho}\mathbf{M} \rightleftarrows \operatorname{Ho}\mathbf{N}$. *Furthermore, the derived pair is an equivalence if and only if the Quillen pair is a Quillen equivalence.*

*Proof.* See [DHK] or [Hov99, 1.3.10 and 1.3.13]. □

Recall that given an object $X$ in a model category $\mathbf{M}$ the categories $X\backslash\mathbf{M}$ and $\mathbf{M}/X$ of objects under and over $X$ are naturally equipped with model category structures, in which the fibrations, cofibrations, and weak equivalences are inherited from $\mathbf{M}$. Furthermore, given a map $f\colon X \to Y$ in $\mathbf{M}$, the induced adjoint functor pairs

$$Y \amalg_X -\colon X\backslash\mathbf{M} \rightleftarrows Y\backslash\mathbf{M} : f^* \qquad \text{and} \qquad f_*\colon \mathbf{M}/X \rightleftarrows \mathbf{M}/Y : X \times_Y -$$

are Quillen pairs. We note that

**Proposition 2.5.** *Let* $\mathbf{M}$ *be a model category, and suppose* $f\colon X \to Y \in \mathbf{M}$. *Then the following are equivalent:*

(1) *The pair* $X\backslash\mathbf{M} \rightleftarrows Y\backslash\mathbf{M}$ *(resp.* $\mathbf{M}/X \rightleftarrows \mathbf{M}/Y$*) is a Quillen equivalence.*
(2) *The pushout (resp. pullback) of* $f$ *along any cofibration (resp. fibration) in* $\mathbf{M}$ *is a weak equivalence.*

*A necessary condition for (1) and (2) to hold is that* $f$ *be a weak equivalence. Sufficient conditions for (1) and (2) to hold are: that* $f$ *be a trivial cofibration (resp. trivial fibration),* or *that* $X$ *and* $Y$ *be cofibrant (resp. fibrant) objects.*

*Proof.* We give the proof of the cofibration case, as the fibration case is strictly dual. Let $i\colon X \to X'$ and $j\colon Y \to Y'$. Then a map $g\colon X' \to Y' \in X\backslash\mathbf{M}$ and its adjoint $g'\colon X' \cup_X Y \to Y' \in Y\backslash\mathbf{M}$ are related by $g = g'f'$, where $f'$ is the pushout of $f$ along $i$. If (1) holds and if $i$ is a cofibration, we can construct $g'$ so that it is a weak equivalence to a fibrant object, and it then follows from (1) that $g$ and hence $f'$ are weak equivalences, giving (2). Conversely, if (2) holds, then $g$ is a weak equivalence if and only if $g'$ is, giving (1).

The necessary condition follows from considering the case when $i$ is the identity map. That $f$ being a trivial cofibration is sufficient is clear; that $X$ and $Y$ being cofibrant is sufficient then follows using (2.6) and the fact that Quillen equivalences satisfy a 2 out of 3 property [Hov99, 1.3.15]. □

**Lemma 2.6.** *Let* $f\colon X \to Y \in \mathbf{M}$ *be a weak equivalence between cofibrant objects. Then the exists a factorizaton* $f = pi$ *such that* $p$ *admits a section* $s$ *and both* $i$ *and* $s$ *are trivial cofibrations.*

*Proof.* See [Hov99, 1.1.12]. □

Thus one has the following reformulation of the notion of properness.

**Proposition 2.7.** *A model category* $\mathbf{M}$ *is left (resp. right) proper if and only if for every weak equivalence* $f\colon X \to Y$ *in* $\mathbf{M}$*, the induced adjoint functor pair* $X\backslash\mathbf{M} \rightleftarrows Y\backslash\mathbf{M}$ *(resp.* $\mathbf{M}/X \rightleftarrows \mathbf{M}/Y$*) is a Quillen equivalence.*

*Remark* 2.8.

(i) If all objects in $\mathbf{M}$ are cofibrant (resp. fibrant) then $\mathbf{M}$ is left (resp. right) proper, because of the sufficiency condition of (2.5).
(ii) Note that if $\mathbf{M}$ is left or right proper, then so are all comma categories $X\backslash\mathbf{M}$ and $\mathbf{M}/X$.

We should note that proper model categories have good theories of homotopy cartesian and cocartesian squares; this topic is treated in detail in [GJ99, II.8].

2.9. **Examples of proper model categories.** The categories of simplicial sets and of topological spaces are examples of proper model categories; for simplicial sets a proof is given in [GJ99, II.8.6]. We will observe in (7.2) that all categories of simplicial algebras are *right* proper.

In certain cases, model categories of simplical algebras (as defined in §7) are known to be left proper (and hence proper). These examples include simplical objects in: all abelian categories, commutative monoids, monoids, simplicial categories with a fixed set of objects [DK80], commutative algebras over any commutative ring $R$, and associative algebras over a field.

2.10. **Examples of improper model categories.** Not every category of simplicial algebras is *left* proper. We offer two examples in which left properness fails. In both cases, left properness is shown to fail by observing that it fails in the simplest case: the functor which takes an simplicial algebra $X$ to the coproduct of $X$ with a free algebra on one generator does not in general take weak equivalences to weak equivalences (cf. 9.1 and 9.2). It should be apparent that many other such examples could be constructed, and that failure of left properness is a "generic" property of categories of simplicial algebras.

*Example* 2.11. Let $T$ be the theory of associative algebras over a commutative ring $R$. If $R$ has Tor-dimension greater than 0, then simplicial $T$-algebras is not a left proper model category. (This example was pointed out to me by Paul Goerss.)

If $A$ is an associative $R$-algebra, the algebra $A\langle x\rangle$ obtained by adjoining one free generator has the form

$$A\langle x\rangle = \bigoplus_{n\geq 1} A^{\otimes n},$$

where the tensor product is taken over $R$. (The $n$-fold tensor product in this sum corresponds to all expressions in $A\langle x\rangle$ of the form $a_1xa_2xa_3x\dots xa_n$, with $a_i \in A$.) Any $R$-algebra $A$ is weakly equivalent to a simplicial $R$-module $B$ which is degreewise flat over $R$, by taking a free resolution. Thus, if there exists an algebra $A$ such that $\mathrm{Tor}_i^R(A, A) \neq 0$ for some $i > 0$ (e.g., $A = R \oplus M \oplus N$ with $\mathrm{Tor}_1^R(M, N) \neq 0$ and with trivial product on $M \oplus N$), then $A\langle x\rangle$ is not weakly equivalent to $B\langle x\rangle$.

*Example* 2.12. Let $C$ denote the theory of augmented commutative $R$-algebras. Any such algebra $A$ has an augmentation ideal $I(A)$. Thus, this category is equivalent to the category of non-unital commutative $R$-algebras, by the functor sending $A \mapsto I(A)$. Let $C_n$ for $n \geq 1$ denote the theory of augmented commutative $R$-algebras with the additional property that $I(A)^n = 0$. The category of simplicial algebras over $C_n$ is not proper for $n \geq 3$.

We give the proof in the case $n = 3$; the general case is no more difficult. In this case, if $A$ is a $C_3$-algebra with augmentation ideal $I = I(A)$, and $A\langle x\rangle$ is the $C_3$-algebra obtained by adjoining one free generator to $A$, we have

$$A\langle x\rangle \approx A[x]/(I, x)^3 \approx A \oplus (A/I^2)x \oplus (A/I)x^2.$$

Since $A/I = R$ and $A/I^2 = R \oplus I/I^2$, the functor $A \mapsto A\langle x\rangle$ is a direct sum of the identity functor, two copies of the functor with constant value $R$, and the indecomposables functor $I \mapsto I/I^2$. The indecomposables functor on $C_3$-algebras has non-trivial higher derived functors, and hence if $B$ is a free simplicial resolution of $A$, then $A\langle x\rangle$ will not in general be weakly equivalent to $B\langle x\rangle$. (A specific example where this occurs is $A = R[y]/y^2$.)

We also note for the record that there are examples of model categories which are not right proper; the first was given by Quillen [Qui69, II.2.9]. Here is a more typical example. Consider the category of simplicial sets, equipped with a model category structure in which weak equivalences are rational

homology isomorphisms, and cofibrations are inclusions; this is an example of Bousfield's localization model category structure [Bou75]. Then one can form a pull-back square of the form

$$\begin{tikzcd} K(\mathbb{Q}/\mathbb{Z},0) \arrow[r, "g"'] \arrow[d] & C \arrow[d, "p"] \\ K(\mathbb{Z},1) \arrow[r, "f"] & K(\mathbb{Q},1) \end{tikzcd}$$

in which $p$ is a rational fibration from a contractible space $C$ and $f$ is a rational homology isomorphism, but $g$ is not a rational homology isomorphism.

## 3. Functors on sets

In this section we characterize functors between categories of sets (and more generally, graded sets) which commute with filtered colimits. This is a prerequisite to our approach to theories in §4.

3.1. **Reflexive coequalizers.** A **reflexive pair** in a category is a diagram consisting of a pair of maps $f, g\colon X \to Y$ together with a map $s\colon Y \to X$ (called a **reflection**) such that $fs = 1_Y = gs$. The colimit of such a diagram is the same as the coequalizer of the pair $f, g$; we call it a **reflexive coequalizer**. We record the following elementary but useful fact.

**Proposition 3.2.** *In $\mathcal{S}$ (the category of sets), reflexive coequalizers commute with finite products.*

3.3. **Functors from finite sets.** Let $f\mathcal{S} \subset \mathcal{S}$ denote a fixed skeleton of the full subcategory of finite sets. It will be convenient to identify $\mathrm{ob} f\mathcal{S}$ with $\mathbb{N}$, and to write $n = \{1, \ldots, n\} \in \mathrm{ob} f\mathcal{S}$ for the distinguished copy of the $n$-element set. We write $X^n = \hom_{\mathcal{S}}(n, X)$.

Let $r\colon \mathrm{Func}(\mathcal{S},\mathcal{S}) \to \mathcal{S}^{f\mathcal{S}}$ denote the restriction functor; it takes an endofunctor on sets to its restriction $f\mathcal{S} \to \mathcal{S}$. This functor admits a left adjoint $\iota\colon \mathcal{S}^{f\mathcal{S}} \to \mathrm{Func}(\mathcal{S},\mathcal{S})$, which associates to each $A \in \mathcal{S}^{f\mathcal{S}}$ its *left Kan extension* $\iota A\colon \mathcal{S} \to \mathcal{S}$ along the full embedding $f\mathcal{S} \subset \mathcal{S}$. This can be presented as a reflexive coequalizer:

$$\coprod_{p\to q} A(p) \times X^q \rightrightarrows \coprod_n A(n) \times X^n \to (\iota A)(X), \tag{3.4}$$

where $X \in \mathcal{S}$ and $A \in \mathcal{S}^{f\mathcal{S}}$. Because $f\mathcal{S} \subset \mathcal{S}$ is a *full* subcategory, one sees that $A \to r\iota A$ is an isomorphism for all $A$ in $\mathcal{S}^{f\mathcal{S}}$, (that is, $(\iota A)(n) \approx A(n)$) and hence that $\iota$ identifies $\mathcal{S}^{f\mathcal{S}}$ up to equivalence as a full subcategory of the category of all functors.

Let $\mathrm{Func}^{\mathrm{f}}(\mathbf{C},\mathbf{D}) \supset \mathrm{Func}^{\mathrm{fr}}(\mathbf{C},\mathbf{D})$ denote the full subcategories of $\mathrm{Func}(\mathbf{C},\mathbf{D})$ consisting of those functors which commute respectively: with filtered colimits; with filtered colimits and reflexive coequalizers. Note that both subcategories are closed under composition of functors.

**Proposition 3.5.** *There is a factorization $\iota\colon \mathcal{S}^{f\mathcal{S}} \to \mathrm{Func}(\mathcal{S},\mathcal{S})$ into*

$$\mathcal{S}^{f\mathcal{S}} \xrightarrow{\iota_1} \mathrm{Func}^{\mathrm{fr}}(\mathcal{S},\mathcal{S}) \xrightarrow{\iota_2} \mathrm{Func}^{\mathrm{f}}(\mathcal{S},\mathcal{S}) \subset \mathrm{Func}(\mathcal{S},\mathcal{S}),$$

*and $\iota_1$ and $\iota_2$ are equivalences of categories.*

*Proof.* Let $A \in \mathcal{S}^{f\mathcal{S}}$. The formula (3.4) for $\iota A$ given above shows that $\iota A(X)$ is computed from $A$ and $X$ using only colimits and finite products, and both of these commute with filtered colimits and reflexive coequalizers. Hence $\iota$ factors through a functor $\iota_1$.

To show that $\iota_1$ and $\iota_2$ are equivalences, it suffices to show that if $F \in \mathrm{Func}^{\mathrm{f}}(\mathcal{S},\mathcal{S})$, then $\eta_F\colon \iota r F \to F$ is an isomorphism. In fact, $\eta_F$ is clearly an isomorphism when evaluated at any finite set, and the result follows from the fact that every set is a filtered colimit of its finite subsets. ∎

As an immediate consequence of (3.5) we see that $\mathcal{S}^{f\mathcal{S}}$ admits the structure of a monoidal category, which corresponds via $\iota$ to composition of functors. We will denote this monoidal structure by $A \circ B$, for $A, B \in \mathcal{S}^{f\mathcal{S}}$, so that $\iota(A \circ B) \approx \iota A \circ \iota B$. The unit corresponds to $I$, defined by $I(n) = \hom_{f\mathcal{S}}(1, n) = n$. Given $A, B \in \mathcal{S}^{f\mathcal{S}}$ and $m \in f\mathcal{S}$, a formula for $(A \circ B)(m)$ can be derived by inserting $B(m)$ for $X$ in (3.4).

3.6. **Graded sets.** Let $\mathcal{I}$ be a set. An **$\mathcal{I}$-graded set** is a collection $(X_i)_{i\in\mathcal{I}}$ of sets, and a morphism of such is collection of maps respecting the grading. The category of $\mathcal{I}$-graded sets is denoted $\mathcal{S}^{\mathcal{I}}$. An $\mathcal{I}$-graded set is said to be **finite** if $\coprod_i X_i$ is a finite set.

We write $f\mathcal{S}/\mathcal{I}$ for the category whose objects are functions $f\colon n \to \mathcal{I}$, $n \in f\mathcal{S}$, and whose morphisms are commuting triangles. A function $f\colon X \to \mathcal{I}$ of sets naturally gives rise to an $\mathcal{I}$-graded set $(f^{-1}(i))_{i\in\mathcal{I}}$, giving an inclusion functor $f\mathcal{S}/\mathcal{I} \to \mathcal{S}^{\mathcal{I}}$ which is equivalent to the inclusion of the full subcategory of *finite* $\mathcal{I}$-graded sets.

Given sets $\mathcal{I}$ and $\mathcal{J}$, let $f\mathcal{S}(\mathcal{I}, \mathcal{J}) = \mathcal{J} \times f\mathcal{S}/\mathcal{I}$. Objects in this category are pairs $(j \in \mathcal{J}, f\colon n \to \mathcal{I})$; a morphism $(j, f) \to (j', f')$ is defined only if $j = j'$, in which case it consists of a map $f \to f' \in f\mathcal{S}/\mathcal{I}$. We write $\mathbb{N}(\mathcal{I}, \mathcal{J}) = \mathrm{ob} f\mathcal{S}(\mathcal{I}, \mathcal{J})$. Then $\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})} = (\mathcal{S}^{\mathcal{J}})^{f\mathcal{S}/\mathcal{I}}$ is equivalent to the category of functors from *finite* $\mathcal{I}$-graded sets to $\mathcal{J}$-graded sets, giving rise to a restriction functor $r\colon \mathrm{Func}(\mathcal{S}^{\mathcal{I}}, \mathcal{S}^{\mathcal{J}}) \to \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$. This functor admits a left adjoint $\iota\colon \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})} \to \mathrm{Func}(\mathcal{S}^{\mathcal{I}}, \mathcal{S}^{\mathcal{J}})$, which associates to each $A \in \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})} = (\mathcal{S}^{\mathcal{J}})^{f\mathcal{S}/\mathcal{I}}$ its *left Kan extension* $\iota A\colon \mathcal{S}^{\mathcal{I}} \to \mathcal{S}^{\mathcal{J}}$ along the full embedding $f\mathcal{S}/\mathcal{I} \subset \mathcal{S}^{\mathcal{I}}$. There is a reflexive coequalizer formula:

$$\coprod_{p\to q\in f\mathcal{S}/\mathcal{I}} A(j,p) \times X^q \rightrightarrows \coprod_{f\in \mathrm{ob} f\mathcal{S}/\mathcal{I}} A(j,f) \times X^f \to (\iota A)(X)_j, \qquad j \in \mathcal{J}, \tag{3.7}$$

where for $f\colon n \to \mathcal{I} \in \mathrm{ob} f\mathcal{S}/\mathcal{I} \subset \mathcal{S}^{\mathcal{I}}$ and $X \in \mathcal{S}^{\mathcal{I}}$ we write $X^f = \hom_{\mathcal{S}^{\mathcal{I}}}(f, X) = \prod_{k\in n} X_{f(k)} \in \mathcal{S}$.

Since $f\mathcal{S}/\mathcal{I} \to \mathcal{S}^{\mathcal{I}}$ is full, we have that $A \approx r\iota A$, so that $(\iota A)(K)_j \approx A(j, K)$ for $j \in \mathcal{J}$ and $K \in f\mathcal{S}/\mathcal{I} \subset \mathcal{S}^{\mathcal{I}}$. We take advantage of this fact to write $A(K) \in \mathcal{S}^{\mathcal{J}}$ for the $\mathcal{J}$-graded set $A(-, K)$.

Considerations identical to those which gave (3.5) give

**Proposition 3.8.** *There is a factorization* $\iota\colon \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})} \to \mathrm{Func}(\mathcal{S}^{\mathcal{I}}, \mathcal{S}^{\mathcal{J}})$ *into*

$$\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})} \xrightarrow{\iota_1} \mathrm{Func}^{\mathrm{fr}}(\mathcal{S}^{\mathcal{I}}, \mathcal{S}^{\mathcal{J}}) \xrightarrow{\iota_2} \mathrm{Func}^{\mathrm{f}}(\mathcal{S}^{\mathcal{I}}, \mathcal{S}^{\mathcal{J}}) \subset \mathrm{Func}(\mathcal{S}^{\mathcal{I}}, \mathcal{S}^{\mathcal{J}}),$$

*and $\iota_1$ and $\iota_2$ are equivalences of categories.*

An immediate consequence of (3.8) is the existence of pairings $-\circ-\colon \mathcal{S}^{f\mathcal{S}(\mathcal{J},\mathcal{K})} \times \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})} \to \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{K})}$ which correspond via $\iota$ to composition of functors. Let $f\mathcal{S}(\mathcal{I}) = f\mathcal{S}(\mathcal{I}, \mathcal{I})$ and $\mathbb{N}(\mathcal{I}) = \mathrm{ob} f\mathcal{S}(\mathcal{I})$. Then $\mathcal{S}^{f\mathcal{S}(\mathcal{I})}$ is a monoidal category, and in fact is a full monoidal subcategory of the category of endofunctors of $\mathcal{S}^{\mathcal{I}}$. The unit object $I$ is defined by $I(i, f\colon n \to \mathcal{I}) = f^{-1}(i)$.

3.9. **Free series.** Let $\mathcal{I}$ and $\mathcal{J}$ be sets. Recall that $\mathbb{N}(\mathcal{I}, \mathcal{J}) = \mathrm{ob} f\mathcal{S}(\mathcal{I}, \mathcal{J})$. The forgetful functor $\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})} \to \mathcal{S}^{\mathbb{N}(\mathcal{I},\mathcal{J})}$ admits a left adjoint $\mathcal{S}\colon \mathcal{S}^{\mathbb{N}(\mathcal{I},\mathcal{J})} \to \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ called the **free series** functor. One easily checks the formula

$$\mathcal{S}A \approx \coprod_{K\in\mathbb{N}(\mathcal{I},\mathcal{J})} A(K) \times I^K,$$

where $I^K \in \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ is defined by $L \mapsto I^K(L) = \hom_{f\mathcal{S}(\mathcal{I},\mathcal{J})}(K, L)$. For $X \in \mathcal{S}^{\mathcal{I}}$ we have

$$(\iota\mathcal{S}A)(X)_j \approx \coprod_{f\colon n\to\mathcal{I}\in f\mathcal{S}/\mathcal{I}} A(f)_j \times X^f,$$

where $X^f \in \mathcal{S}$ is as in (3.7). In the case when $\mathcal{I} = \mathcal{J}$ is a singleton, these formulas reduce to $\mathcal{S}A \approx \coprod_n A(n) \times I^n$ and $(\iota\mathcal{S}A)(X) = \coprod_n A(n) \times X^n$.

3.10. **Simplicial objects.** By prolongation, we obtain a functor $s\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})} \to \mathrm{Func}(s\mathcal{S}^{\mathcal{I}}, s\mathcal{S}^{\mathcal{J}})$. The image of this functor is the full subcategory of *simplicial objects* in the category of functors which commute with filtered colimits and reflexive coequalizers. Formulas (3.4) and (3.7) still apply in this case, where the objects are now *graded* simplicial sets.

## 4. Theories, algebras, and bimodules

In this section, we define algebraic theories and their associated algebra categories. In our approach, we also consider multi-sorted theories. We also give some attention to *bimodules* of theories, which give rise to a large class of functors between categories of algebras, and will play an important role in §§8 and 11. The definitions of theories and algebras that we give appear quite different than the notions of algebraic theories and their models as in [Law63], where a theory is defined to be a category with finite products (see the nice treatment in [Bor94, Ch. 3] for this). However, our categories of "algebras" are the same as the categories of "models", as we note below (4.2). Our formulation is one of those used by Boardman and Vogt in a topological context [BV73] (they write "theories with colours" for what we call "multi-sorted theories"). It is also close to that given by Schwede [Sch], although the theories he considers are *pointed*. Lawvere's original formulation is also used in [BV73], and is used in a crucial way by Badzioch [Bad00].

In what follows, we make repeated use of the identifications of $\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ and $s\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ as full subcategories of the respective functor categories, and we omit use of the $\iota$ symbol of §3; thus we write $A(X)$ where before we had $(\iota A)(X)$.

4.1. **Theories.** Let $\mathcal{I}$ be a set, and recall that $\mathcal{S}^{f\mathcal{S}(\mathcal{I})}$ is a monoidal category, equivalent to a full monoidal subcategory of $\mathrm{Func}(\mathcal{S}^{\mathcal{I}}, \mathcal{S}^{\mathcal{I}})$. We define an **$\mathcal{I}$-sorted theory**, or more simply a **theory**, to be a monoid object $T$ in $\mathcal{S}^{f\mathcal{S}(\mathcal{I})}$. That is, $T \in \mathcal{S}^{f\mathcal{S}(\mathcal{I})}$ is equipped with maps $\mu_T\colon T \circ T \to T$ and $\eta_T\colon I \to T$ satisfying the usual axioms for a monoid. From (3.8), we see that $\mathcal{I}$-sorted theories are essentially the same as monads on $\mathcal{S}^{\mathcal{I}}$ which commute with filtered colimits.

We write $\mathcal{T}(\mathcal{I})$ for the category of $\mathcal{I}$-sorted theories over sets.

4.2. **Algebras over a theory.** An **algebra** $X$ over an $\mathcal{I}$-sorted theory $T$ is an algebra over the monad induced by $T$; that is, an algebra is an object $X \in \mathcal{S}^{\mathcal{I}}$ equipped with a map $\psi\colon T(X) \to X$ satisfying the usual axioms. The category of $T$-algebras is denoted $T$-alg.

Given a graded set $X$, the object $T(X)$ is naturally a $T$-algebra, namely the **free $T$-algebra on $X$**.

**Proposition 4.3.** *Let $T$ be an $\mathcal{I}$-sorted theory. The category $T$-alg is complete and cocomplete. Limits, filtered colimits and reflexive coequalizers are created in the underlying category $\mathcal{S}^{\mathcal{I}}$. There exists an adjoint functor pair $T\colon \mathcal{S}^{\mathcal{I}} \rightleftarrows T\text{-alg} : u$, where $u$ is the forgetful functor, and $T$ is called the free $T$-algebra functor.*

*Proof.* That limits, filtered colimits, and reflexive coequalizers exist and are created in $\mathcal{S}^{\mathcal{I}}$ is immediate from (3.8). That the free algebra functor is left adjoint is a standard property of monads [Bor94, 4.1.4]. Existence of colimits follows from [Bor94, 4.3.6]; or note that colimits of a diagram $\alpha \mapsto X_\alpha\colon \mathbf{A} \to T$-alg can be constructed explicitly as the reflexive coequalizer in $T$-alg of $T(\mathrm{colim}^{\mathcal{S}^{\mathcal{I}}}_{\mathbf{A}}\, TX_\alpha) \rightrightarrows T(\mathrm{colim}^{\mathcal{S}^{\mathcal{I}}}_{\mathbf{A}}\, X_\alpha)$, the top map being induced by the inclusions $X_\alpha \to \mathrm{colim}^{\mathcal{S}^{\mathcal{I}}}_{\mathbf{A}}\, TX_\alpha$ and the top map being induced by the algebra structure maps $TX_\alpha \to X_\alpha$. ∎

According to our definition, a single-sorted theory $T$ corresponds, via (3.5), precisely to a monad on sets which commutes with filtered colimits. By [Bor94, 4.6.2], categories of algebras over such monads (which we have called "algebras over a theory") are exactly those which are equivalent to categories of "models of an algebraic theory" in the classical sense. See also [BV73, Prop. 2.30].

Finally, note that the category $\mathcal{T}(\mathcal{I})$ of $\mathcal{I}$-sorted theories is itself an example of a category of algebras over a certain $\mathbb{N}(\mathcal{I})$-sorted theory, namely the theory of $\mathcal{I}$-sorted theories. This is because the forgetful functor $\mathcal{T}(\mathcal{I}) \to \mathcal{S}^{\mathbb{N}(\mathcal{I})}$ admits a left adjoint, and is monadic. Thus (4.3) shows that the category of such theories is complete and cocomplete, and that there exist free theories. We will consider an explicit construction of free theories in §10.

4.4. **Bimodules.** Given $S \in \mathcal{T}(\mathcal{I})$ and $T \in \mathcal{T}(\mathcal{J})$, a **$T,S$-bimodule** is an object $M \in \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ equipped with actions $T \circ M \to M$ and $M \circ S \to M$, which are associative and unital and which commute with each other. Let $T,S$-mod denote the category of bimodules. A **right $S$-module** is an $I,S$-bimodule and a **left $T$-module** is a $T,I$-bimodule.

Given an $S$-algebra $X$, let $M \circ_S X$ denote the coequalizer of the following reflexive pair in $T$-alg (which can be computed in graded sets by (4.3)):

$$M(S(X)) \rightrightarrows M(X) \to M \circ_S X.$$

This gives rise to a functor $\iota\colon T,S\text{-mod} \to \mathrm{Func}(S\text{-alg}, T\text{-alg})$. (Warning: this is not the $\iota$ used in §3.) Note that if $K \in f\mathcal{S}/\mathcal{I} \subset \mathcal{S}^{\mathcal{I}}$, then $M \circ_S S(K) \approx M(K)$ as objects of $\mathcal{S}^{\mathcal{J}}$; that is, $M(K)$ is the value of $M \circ_S -$ on the free $S$-algebra generated by $K$.

**Proposition 4.5.** *Let $S$ and $T$ be $\mathcal{I}$- and $\mathcal{J}$-sorted theories over sets. The functor $\iota\colon T,S\text{-mod} \to \mathrm{Func}(S\text{-alg}, T\text{-alg})$ defined above factors through, and induces an equivalence with, the full subcategory $\mathrm{Func}^{\mathrm{fr}}(S\text{-alg}, T\text{-alg})$ of functors which commute with filtered colimits and reflexive coequalizers.*

*Proof.* It is clear using (3.8) that $\iota$ factors through the subcategory. It remains to show that $\iota$ is an equivalence.

Let $S\text{-alg}^{\mathrm{fgf}} \subset S\text{-alg}$ denote the full subcategory of *finitely generated free* $S$-algebras; every object in this subcategory is isomorphic to $S(K)$ for some $K \in f\mathcal{S}/\mathcal{I}$. Consider the sequence of functors

$$T,S\text{-mod} \xrightarrow{\iota} \mathrm{Func}^{\mathrm{fr}}(S\text{-alg}, T\text{-alg}) \subset \mathrm{Func}(S\text{-alg}, T\text{-alg}) \to \mathrm{Func}(S\text{-alg}^{\mathrm{fgf}}, T\text{-alg});$$

the right-hand arrow is the one induced by restriction of functors to the subcategory. The result will follow when we show that the composites $\alpha\colon \mathrm{Func}^{\mathrm{fr}}(S\text{-alg}, T\text{-alg}) \to \mathrm{Func}(S\text{-alg}^{\mathrm{fgf}}, T\text{-alg})$ and $\beta\colon T,S\text{-mod} \to \mathrm{Func}(S\text{-alg}^{\mathrm{fgf}}, T\text{-alg})$ are equivalences.

To see that $\alpha$ is an equivalence, observe that every $S$-algebra is a coequalizer of a reflexive diagram of free algebras, and that every free $S$-algebra is a filtered colimit of finitely generated free algebras. Thus every functor $S\text{-alg} \to T\text{-alg}$ which commutes with filtered colimits and reflexive coequalizers is determined up to unique isomorphism by its restriction to the subcategory of finitely generated free algebras, and natural transformation between such functors are uniquely determined by this restriction. Any functor $S\text{-alg}^{\mathrm{fgf}} \to T\text{-alg}$ extends to an element of $\mathrm{Func}^{\mathrm{fr}}(S\text{-alg}, T\text{-alg})$ by a left Kan extension construction, and therefore this construction gives the inverse to $\alpha$.

We now show that $\beta$ is an equivalence. Explicitly, $\beta$ sends $M \in T,S\text{-mod}$ to the functor $G\colon X \mapsto M \circ_S X$; note that if $X \approx S(K)$, then $G(X) \approx M(K)$. We will construct an inverse $\gamma\colon \mathrm{Func}(S\text{-alg}^{\mathrm{fgf}}, T\text{-alg}) \to T,S\text{-mod}$. Given $F\colon S\text{-alg}^{\mathrm{fgf}} \to T\text{-alg}$, define $N \in \mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ by $N(K) = F(S(K))$; recall that under the equivalence $\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})} \approx \mathrm{Func}^{\mathrm{fr}}(\mathcal{S}^{\mathcal{I}}, \mathcal{S}^{\mathcal{J}})$, the object $N$ corresponds to a functor $\mathcal{S}^{\mathcal{I}} \to \mathcal{S}^{\mathcal{J}}$ also denoted by $N$, and there is a map $N(X) \to F(S(X))$ natural in $X \in \mathcal{S}^{\mathcal{I}}$. Give $N$ the structure of a left $T$-module by

$$(T \circ N)(K) = T(F(S(K))) \to F(S(K)) = N(K),$$

using the fact that $F$ takes values in $T$-algebras. Give $N$ the structure of a right $S$-module by

$$(N \circ S)(K) \approx N(S(K)) \to F(S(S(K))) \xrightarrow{F(\mu_K)} F(S(K)),$$

using the $S$-algebra structure of $S(K)$. It follows that $N$ is a $T,S$-bimodule, that our construction $F \mapsto \gamma F = N$ is a functor from functors to bimodules, and that $\beta\gamma \approx \mathrm{id}$ and $\gamma\beta \approx \mathrm{id}$ as desired.

□

*Remark* 4.6. Let $S \in \mathcal{T}(\mathcal{I})$ and $T \in \mathcal{T}(\mathcal{J})$. Then the category $T, S$-mod is a category of algebras over a certain $\mathbb{N}(\mathcal{I}, \mathcal{J})$-sorted theory $B_{T,S}$; this is because bimodules are simply algebras over the monad $A \mapsto T \circ A \circ S$ on $\mathbb{N}(\mathcal{I}, \mathcal{J})$-graded sets, which commutes with filtered colimits.

Suppose that $X \in S$-alg, and consider the functor $T, S$-mod $\to T$-alg given by $M \mapsto M \circ_S X$. This functor commutes with filtered colimits and reflexive coequalizers by construction, and thus by (4.5) it is represented by a certain $T, B_{T,S}$-bimodule $N_X$ (whose underlying set is graded by $\mathbb{N}(\mathbb{N}(\mathcal{I}, \mathcal{J}), \mathcal{J})$!) We will need this observation in §8.

Given a morphism $\varphi\colon S \to T$ of $\mathcal{I}$-sorted theories, there is an evident restriction functor $\varphi^*\colon T\text{-alg} \to S\text{-alg}$, which is the identity on underlying graded sets.

**Proposition 4.7.** *The restriction functor $\varphi^*$ admits a left adjoint functor $\varphi_*\colon S\text{-alg} \to T\text{-alg}$, and $\varphi_* X \approx T \circ_S X$.*

*Proof.* Let $Y \in T$-alg. Then $\hom_{T\text{-alg}}(\varphi_* X, Y)$ is the equalizer of $\hom_{T\text{-alg}}(TX, Y) \rightrightarrows \hom_{T\text{-alg}}(TSX, Y)$, or equivalently of $\hom_{\mathcal{S}^{\mathcal{I}}}(X, Y) \rightrightarrows \hom_{\mathcal{S}^{\mathcal{I}}}(SX, Y)$, where the two arrows send $f\colon X \to Y$ to $f(\psi_X)$ and $(\psi_Y)(\varphi X)(Sf)$ respectively, where $\psi_X\colon SX \to X$ and $\psi_Y\colon TY \to Y$ denote the algebra structure maps. □

4.8. **Undercategories and coproducts of theories.** Let $T$ be an $\mathcal{I}$-sorted theory, and $X$ a $T$-algebra. Define $T_X \in \mathcal{S}^{f\mathcal{S}(\mathcal{I})}$ by $T_X(K) = T(K) \amalg^T X$.

**Proposition 4.9.** *The object $T_X$ admits the structure of a theory, and there is an equivalence of categories $T_X$-alg $\approx X \backslash T$-alg.*

*Proof.* Define $I \to T_X$ to be the evident map $I(K) \approx K \to T(K) \amalg^T X \approx T_X(K)$, and $T_X \circ T_X \to T_X$ to be the evident map $(T_X \circ T_X)(K) \approx T_X(T_X(K)) \approx T(T(K) \amalg^T X) \amalg^T X \to T(K) \amalg^T X \approx T_X(K)$. Then $T_X$ is easily seen to be a theory, and the evident functor $T_X$-alg $\to X \backslash T$-alg an equivalence. □

Given $X \in \mathcal{S}^{\mathcal{I}}$ and $g\colon X \to Y \in \mathcal{S}^{\mathcal{I}}$, there exist **endomorphism theories** $\mathcal{E}_X$ and $\mathcal{E}_g$, with the property that $\hom_{\mathcal{T}(\mathcal{I})}(T, \mathcal{E}_X)$ is in bijective correspondence with the set of $T$-algebra structures on $X$, and $\hom_{\mathcal{T}(\mathcal{I})}(T, \mathcal{E}_g)$ is in bijective correspondence with the set of pairs of $T$-algebra structures on $X$ and $Y$ which make $f$ a map of $T$-algebras. They are given by the formulas

$$\mathcal{E}_X(j, f) = \hom_{f\mathcal{S}(\mathcal{I}, \mathcal{J})}(X^f, X_j),$$
$$\mathcal{E}_g(j, f) = \hom_{f\mathcal{S}(\mathcal{I}, \mathcal{J})}(X^f, X_j) \times_{\hom_{f\mathcal{S}(\mathcal{I}, \mathcal{J})}(X^f, Y_j)} \hom_{f\mathcal{S}(\mathcal{I}, \mathcal{J})}(Y^f, Y_j).$$

Here $X^f$ is as in (3.7).

Let $S$ and $T$ be two $\mathcal{I}$-indexed theories. Then $S \amalg^{\mathcal{T}(\mathcal{I})} T$ denotes the coproduct in $\mathcal{T}(\mathcal{I})$.

**Proposition 4.10.** *The category $(S \amalg^{\mathcal{T}(\mathcal{I})} T)$-alg has as objects $X \in \mathcal{S}^{\mathcal{I}}$ equipped with both an $S$-algebra and a $T$-algebra structure, and as morphisms those maps which commute with both algebra structures.*

*Proof.* This is immediate from the existence of the endomorphism theories. □

4.11. **Simplicial objects.** We can similarly consider **simplicial theories**, namely monoid objects in $s\mathcal{S}^{f\mathcal{S}(\mathcal{I})}$; these are the same as simplicial objects in $\mathcal{T}(\mathcal{I})$, and we write $s\mathcal{T}(\mathcal{I})$ for the category of $\mathcal{I}$-simplicial theories.

If $T$ is a simplicial theory, then by a $T$-algebra $X$ we mean a **simplicial algebra**, namely an object of $s\mathcal{S}^{\mathcal{I}}$ which is an algebra over the monad induced by $T$. Effectively, if $T = \{T_n\}$ is a simplicial theory, $X$ amounts to a collection $\{X_n\}$ of objects in $\mathcal{S}^{\mathcal{I}}$, such that each $X_n$ is equipped with the

structure of a $T_n$ algebra, together with, for each $\delta\colon [m] \to [n] \in \Delta$, a map $X_\delta\colon X_n \to (T_\delta)^* X_m$ of $T_n$-algebras, with the conditions $X_{\delta'} X_\delta = X_{\delta\delta'}$.

Similarly, we have **simplicial bimodules**; these are objects $M$ in $s\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ which are $T, S$-bimodules, or equivalently a collection $M = \{M_n\}$ of objects in $\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ such that each $M_n$ is a $T_n, S_n$-bimodule, together with simplicial operators acting as above.

Propositions (4.3), (4.7), (4.9), and (4.10) carry over to the simplicial setting: change $\mathcal{S}$ to $s\mathcal{S}$. There is also a simplicial analogue of (4.5). We say a functor $F\colon S\text{-alg} \to T\text{-alg}$ between categories of simplicial algebras is **degreewise** if there exist functors $F_n\colon S_n\text{-alg} \to T_n\text{-alg}$ such that $F(X)_n \approx F_n(X_n)$ for $n \geq 0$, together with natural transformations $F_\delta(S_\delta)^*\colon F_n \to (T_\delta)^* F_m$ for each $\delta\colon [m] \to [n] \in \Delta$ satisfying the appropriate identities.

**Proposition 4.12.** *Let $S$ and $T$ be $\mathcal{I}$ and $\mathcal{J}$-sorted simplicial theories. Then the functor $\iota\colon T, S\text{-mod} \to \operatorname{Func}(S\text{-alg}, T\text{-alg})$ factors through, and induces an equivalence with, the full subcategory $\operatorname{Func}^{\mathrm{dfr}}(S\text{-alg}, T\text{-alg})$ of degreewise functors which in each degree commute with filtered colimits and reflexive coequalizers.*

*Proof.* Apply (4.5) in each simplicial degree. □

## 5. Functors commuting with products

Henceforward, we consider only *simplicial* $\mathcal{I}$-sorted theories and simplicial algebras over such, unless otherwise indicated.

Let $E\colon s\mathcal{S} \to s\mathcal{S}$ be a functor. Such a functor induces a functor $s\mathcal{S}^{\mathcal{I}} \to s\mathcal{S}^{\mathcal{I}}$, which we also denote by $E$. We say that $E$ **commutes with products** if $E(1) \approx 1$ and for all $X, Y \in s\mathcal{S}$ the natural map $E(X \times Y) \to EX \times EY$ is an isomorphism. Note that if $f\colon p \to n \in f\mathcal{S} \subset \mathcal{S}$ and if we write $X^f\colon X^n \to X^p$ for the induced map on products, then $E(X^f) \approx (EX)^f$.

Suppose that in addition there is a natural transformation $\eta\colon \mathrm{Id} \to E$. Then there exist natural maps $X \times EY \to E(X \times Y)$ for all $X, Y \in s\mathcal{S}$; furthermore, these maps are coherent, in the sense that both ways to obtain a map $X \times Y \times EZ \to E(X \times Y \times Z)$ are the same. In particular, $E$ is a *simplicial* functor.

Let $F = \mathcal{S}A \in s\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ be a free series (3.9) on some $A \in s\mathcal{S}^{\mathbb{N}(\mathcal{I},\mathcal{J})}$, and let $X \in s\mathcal{S}^{\mathcal{I}}$. The above discussion shows that there is an evident map $\alpha\colon F(E(X)) \to E(F(X))$ in $s\mathcal{S}^{\mathcal{J}}$; for instance, in the case when $\mathcal{I} = *$, the map is

$$\alpha\colon F(E(X)) \approx \coprod_n A(n) \times (EX)^n \to \coprod_n E\left(A(n) \times X^n\right) \to E\left(\coprod_n A(n) \times X^n\right) \approx E(F(X)),$$

induced by $(EX)^n \approx E(X^n)$ and $X \times EY \to E(X \times Y)$.

**Proposition 5.1.** *Let $E\colon s\mathcal{S} \to s\mathcal{S}$ be a functor commuting with products and equipped with a natural transformation $\eta\colon \mathrm{Id} \to E$. Then for all $X \in s\mathcal{S}^{\mathcal{I}}$ and $F \in s\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$, there exist maps*

$$\widetilde{\alpha}\colon F(E(X)) \to E(F(X))$$

*which are natural in $X$ and $F$, and which in the case that $F = \mathcal{S}A$ is the map $\alpha$ described above. Furthermore, the $\widetilde{\alpha}$ are the unique collection of maps with this property.*

*Proof.* For convenience, we write the proof only in the case $\mathcal{I}$ and $\mathcal{J}$ are singleton sets; the general case is only notationally more difficult.

We first show that $\alpha$ is in fact a natural transformation between functors defined on the full subcategory of free objects in $s\mathcal{S}^{f\mathcal{S}}$. Consider a map $\mathcal{S}A \to \mathcal{S}B$ between free objects. This amounts to a collection of maps $A(n) \to \coprod_p B(p) \times n^p$, $n \geq 0$, and the induced map $\mathcal{S}A(X) \to \mathcal{S}B(X)$ factors

$$\coprod_n A(n) \times X^n \to \coprod_{n,p} B(p) \times n^p \times X^n \to \coprod_p B(p) \times X^p.$$

To show that $\alpha$ commutes with these maps reduces to showing that it commutes with $n^p \times X^n \to X^p$, which is clear.

Define $\widetilde{\alpha} = \alpha$ on the full subcategory of free objects in $s\mathcal{S}^{f\mathcal{S}}$. Since every object of $s\mathcal{S}^{f\mathcal{S}}$ is a reflexive coequalizer of a pair of free objects, the $\widetilde{\alpha}$ extend in a unique way to arbitrary objects in $s\mathcal{S}^{f\mathcal{S}}$. □

By the uniqueness property of (5.1), we see that $\widetilde{\alpha}\colon \mathrm{Id}(E(X)) \to E(\mathrm{Id}(X))$ is the identity, and the two ways of getting maps $G(F(E(X))) \to E(G(F(X)))$ coincide: $\widetilde{\alpha}_{GF} = (\widetilde{\alpha}_G F)(G\widetilde{\alpha}_F)$.

**Corollary 5.2.** *Let $T$ be a (possibly multi-sorted) simplicial theory. A product preserving functor $E\colon s\mathcal{S} \to s\mathcal{S}$ equipped with a natural transformation $\eta\colon \mathrm{Id} \to E$ lifts in a natural way to a functor $E\colon T\text{-alg} \to T\text{-alg}$. Furthermore, for every $M \in T, S\text{-mod}$ and every $X \in T\text{-alg}$ there exist maps*

$$\widetilde{\alpha}\colon M \circ_T EX \to E(M \circ_S X)$$

*which are natural in $M$ and $X$, and coherent with respect to compositions of functors.*

*Example* 5.3.

(1) Let $K$ be a fixed simplicial set, and let $X^K$ mapping complex from $K$ to $X$. Then $X \mapsto X^K$ commutes with products and there are natural maps $X \to X^K$ induced by the projection $K \to 1$. Then (5.1) says that our functor $F\colon s\mathcal{S}^{\mathcal{I}} \to s\mathcal{S}^{\mathcal{J}}$ is a *simplicial functor*, and (5.2) says that $X^K$ is a $T$-algebra if $X$ is.

(2) Define $E(X) = \mathrm{Sing}|X|$, the singular complex of the geometric realization of the underlying simplicial sets; this commutes with products and admits a natural transformation $\mathrm{Id} \to E$. Then (5.2) says that $E$ lifts to all categories of $T$-algebras.

(3) Similarly, if $E(X) = \mathrm{Ex}^\infty(X)$, the functor of [Kan57], then (5.2) says that $E$ lifts to all categories of $T$-algebras.

## 6. Simplicial algebras and s-free maps

In this section we explain the notion of an *s-free map;* this terminology is due to Goerss and Hopkins [GHa]; it is essentially what Quillen calls a *free* map [Qui67, II.4].

6.1. **Free degeneracy diagrams.** Let $\Delta$ denote the category of finite totally ordered sets of the form $[n] = \{0, \dots, n\}$ and weakly monotone maps between them. The category of simplicial objects in $\mathbf{C}$ is just the category of functors $\Delta^{\mathrm{op}} \to \mathbf{C}$. The **degeneracy category** $\Delta_+$ is the subcategory of $\Delta$ consisting of all *surjective* maps. A **degeneracy diagram** in $\mathbf{C}$ is a functor $F\colon \Delta_+^{\mathrm{op}} \to \mathbf{C}$.

A degeneracy diagram $K\colon \Delta_+^{\mathrm{op}} \to \mathcal{S}$ is **free** if there exist sets $L_n \subset K_n$ and an isomorphism of degeneracy diagrams

$$K_n \approx \coprod_{\sigma\colon [m]\to[n]\in\Delta_+} L_m.$$

That is, $K$ is a left Kan extension of $L\colon \mathbb{N}^{\mathrm{op}} \to \mathcal{S}$ along the inclusion $\mathbb{N}^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ sending $n \mapsto [n]$.

It is well known that if $X$ is a simplicial set, then the underlying degeneracy diagram of $X$ is free. More is true. Let $\Delta_0 \subset \Delta$ denote the subcategory consisting of those morphisms $\delta\colon [m] \to [n]$ such that $\delta(0) = 0$. (A functor $\Delta_0 \to \mathbf{C}$ is precisely an augmented simplicial object in $\mathbf{C}$ with a contracting homotopy.) Note that $\Delta_+ \subset \Delta_0$.

**Lemma 6.2.**

(1) *If $X\colon \Delta_0^{\mathrm{op}} \to \mathcal{S}$, then the underlying degeneracy diagram of $X$ is free.*

(2) *Suppose $Y \subset X$ is an inclusion of degeneracy diagrams of sets, and that $X$ free. Then $Y$ is free if and only if for all $x \in X_n$ and $\sigma\colon [m] \to [n] \in \Delta_+$, $\sigma(x) \in Y_m$ implies that $x \in Y_n$.*

*Proof.* Let $X$ be as in (1). For this proof, we will write simplicial operators as acting on the right. Say that an $x \in X_n$ in **non-degenerate** if it is not of the form $y\sigma$ for some non-identity $\sigma\colon [n] \to [m] \in \Delta_+$ and some $y \in X_m$. We claim: if $x \in X_k$, $x' \in X_\ell$ are non-degenerate elements such that $x\sigma = x'\sigma' \in X_n$ for some $\sigma, \sigma' \in \Delta_+$, then

(a) $k = \ell$ and $x = x'$, and
(b) $\sigma = \sigma'$.

From this claim it will follow that for each $y \in X_n$ there is a unique non-degenerate $x \in X_m$ and a unique $\sigma\colon [n] \to [m] \in \Delta_+$ such that $y = x\sigma$; that is, the underlying degeneracy diagram of $X$ is *free* on the non-degenerate elements, proving (1). To prove the claim, observe that there exist $\delta, \delta' \in \Delta_0$ such that $\sigma\delta = \mathrm{id}_{[k]}$ and $\sigma'\delta' = \mathrm{id}_{[\ell]}$. Then $x'\sigma'\delta = x\sigma\delta = x$ and $x\sigma\delta' = x'\sigma'\delta' = x'$. Any map in $\Delta_0$ must factor uniquely in the form $\delta_1\sigma_1$ for an injective $\delta_1$ and surjective $\sigma_1$; this fact applied to $\sigma\delta'$ and $\sigma'\delta$ together with the non-degeneracy of $x$ and $x'$ implies that $\sigma\delta' = \sigma'\delta = \mathrm{id}$ and hence that $x = x'$, proving (a). To get (b), observe that the same argument shows that $\sigma$ and $\sigma'$ must admit exactly the same elements of $\Delta_0$ as right inverses, and it is easy to derive (b) from this.

To show (2) observe that $X$, being free, is a disjoint union of free degeneracy diagrams on one generator (in various degrees), and that a free degeneracy diagram on one generator has no non-trivial free sub-diagrams. ∎

6.3. ***s*-free morphisms.** We say a morphism $f\colon X \to Y \in T$-alg is ***s*-free** if, after restricting from $\Delta$ to the degeneracy category, there is an isomorphism

$$Y \approx X \coprod^{T\text{-alg}} T(K),$$

where $K$ is a free degeneracy diagram in $\mathcal{I}$-graded sets. This means that for each $n \geq 0$, $Y_n \approx X_n \amalg^{T_n} T_n K_n$, and the $K_n$'s are closed under degeneracy operations. (The complication here is that each level $Y_n$ in the simplicial algebra is an object in a *different* category for each $n$).

An object $X \in T$-alg is said to be ***s*-free** if the map from the initial object to $X$ is $s$-free. Note that $f\colon X \to Y \in T$-alg is an $s$-free morphism if and only if $Y$ is an $s$-free object in the comma category $X\backslash T$-alg $\approx T_X$-alg.

**Proposition 6.4.** *Let $X$ be a simplicial $T$-algebra, and define a simplicial object $Y$ in* $T$-alg *by* $[n] \mapsto Y_{n,*} \approx T^{n+1}X$. *Then* $\mathrm{diag}(Y) \in T$-alg *is $s$-free.*

*Proof.* We have that $Y_{n,n} \approx (T_n)^{n+1}X_n \approx T_n((T_n)^n X_n)$; thus, we must show that $[n] \mapsto (T_n)^n X_n$ is a free degeneracy diagram of $\mathcal{I}$-graded sets. First, suppose that $T$ and $X$ are a *discrete* theory and algebra. The degeneracy diagram $\Delta_+ \to \mathcal{S}^{\mathcal{I}}\colon [n] \mapsto T^n X$ extends to a functor $\Delta_0 \to \mathcal{S}^{\mathcal{I}}$, using the fact that $T$ is a monad and and $X$ and algebra: the "face" maps are given by $T^i \mu_T T^{n-i-2}\colon T^n X \to T^{n-1}X$ and $T^{n-1}\psi_X\colon T^n X \to T^{n-1}X$.

Since the extension from a $\Delta_+$-diagram to a $\Delta_0$-diagram is natural in $T$ and $X$, we see that $[n] \mapsto (T_n)^n X_n$ is the "diagonal" of a *simplicial* object in $\Delta_0$-diagrams, and in particular it is a $\Delta_0$-diagram, and the result follows using (6.2) (1). ∎

## 7. Homotopy theory of algebras

In this section we describe a model category structure on the category of simplicial algebras over any $\mathcal{I}$-sorted theory $T$ based on simplicial sets. The model category structure we construct coincides with those constructed in [Sch] and [Bad00].

7.1. **Closed model category structure.** Let $T$ be an $\mathcal{I}$-sorted theory over $s\mathcal{S}$. Recall that there is a forgetful functor $T$-alg $\to s\mathcal{S}^{\mathcal{I}}$. Write $U_\alpha\colon T$-alg $\to s\mathcal{S}$ for the underlying simplicial set corresponding to $\alpha \in \mathcal{I}$.

We say that a morphism $f\colon X \to Y$ is a **strong retract** of $g\colon X \to Y'$ if $f$ is a retract of $g$ in the category of objects under $X$.

**Theorem 7.2.** *The category $T$-alg admits a simplicial model category structure in which $f\colon X \to Y \in T$-alg is*

(1) *a fibration or a weak equivalence if and only if each $U_\alpha(f), \alpha \in \mathcal{I}$ is a fibration or weak equivalence of simplicial sets, and*
(2) *a cofibration if and only if it is a strong retract of an $s$-free map.*

*Furthermore, this model category is right proper.*

Let $\varphi\colon S \to T \in s\mathcal{T}(\mathcal{I})$ be a morphism of $\mathcal{I}$-sorted simplicial theories.

**Corollary 7.3.** *The induced adjoint pair $\varphi_*\colon S\text{-alg} \rightleftarrows T\text{-alg} : \varphi^*$ (4.7) is a Quillen pair between the corresponding model categories.*

*Proof.* The right adjoint $\varphi^*$ is the identity on the underlying simplicial sets, and hence preserves weak equivalences and fibrations, and thus the left adjoint preserves cofibrations. □

*Example* 7.4. The categories $s\mathcal{S}^{\mathcal{I}}$ of graded simplicial sets admit a model category structure in which a map is a fibration, cofibration, or weak equivalence if it is such in each $\mathcal{I}$-grading.

*Example* 7.5. The category $s\mathcal{T}$ of simplicial theories is a category of algebras over an $\mathbb{N}$-sorted theory, and so admits a simplicial model category structure; similarly for categories of bimodules over such theories. More generally, the category $s\mathcal{T}(\mathcal{I})$ of $\mathcal{I}$-sorted simplicial theories admits a simplicial model category structure, as do categories of bimodules over simplicial multi-sorted theories.

We will only sketch the proof of (7.2); it is an instance of the "small object argument", which was already used by Quillen [Qui67] for the case of simplicial algebras over a discrete theory. (A more recent exposition of Quillen's proof for simplicial algebras is [GJ99, II.5].) We note that the statement about right properness follows from the fact that pullbacks, fibrations, and weak equivalences are created by the $U_\alpha$'s, and that $s\mathcal{S}$ is right proper. The fact that $T$-alg is a *simplicial* model category follows by a straightforward argument using (5.2) and (5.3) (1), together with the fact that graded simplicial sets are a simplicial model category.

To apply the small object argument, we must name sets of "generating cofibrations" and "generating trivial cofibrations". In our case we can take as generating cofibrations the set of maps

$$T(K \times \partial\Delta[n]) \to T(K \times \Delta[n]), \qquad K \in \mathrm{ob} f\mathcal{S}/\mathcal{I},\ n \geq 0,$$

and as generating trivial cofibrations the set of maps

$$T(K \times \Lambda^k[n]) \to T(K \times \Delta[n]), \qquad K \in \mathrm{ob} f\mathcal{S}/\mathcal{I},\ n \geq k \geq 0,$$

where $\Delta[n] \supset \partial\Delta[n] \supset \Lambda^k[n]$ are the standard $n$-simplex, its boundary, and its $k$-th "horn". Here we regard $f\mathcal{S}/\mathcal{I} \subset \mathcal{S}^{\mathcal{I}} \subset s\mathcal{S}^{\mathcal{I}}$ as usual, and also $s\mathcal{S} \subset s\mathcal{S}^{\mathcal{I}}$ by the diagonal inclusion. (We really only need to use those $K$ whose underlying set is a singleton.)

Using the small object argument, it is straightforward to produce factorizations (map) = (triv. fib)($s$-free). To get factorizations (map) = (fib.)(triv. cof.) we need the following lemma, which ensures that the putative trivial cofibrations produced by the small object argument are in fact such.

**Lemma 7.6.** *Suppose $f\colon X \to Y \in T$-alg is a map which has the left lifting property with respect to all fibrations (as defined in (7.2)). Then $f$ is a weak equivalence.*

*Proof.* Let $\gamma\colon \mathrm{Id} \to E$ be a natural transformation of functors $s\mathcal{S} \to s\mathcal{S}$ such that $E$ is product preserving, $E(X)$ is a fibrant simplicial set and $\gamma_X\colon X \to E(X)$ is a weak equivalence for all $X$; we

can use examples (5.3) (2) or (3). This functor $E$ extends to $T$-alg by (5.2). Now consider

$$\begin{tikzcd}
X \arrow[r, "i"'] \arrow[d, "f"] & (EY)^{\Delta[1]} \times_{EY} EX \arrow[r, "\sim", "\pi"'] \arrow[d, "p"'] & EX \\
Y \arrow[r, "j"'] \arrow[ur, dotted] & EY &
\end{tikzcd}$$

where the fiber product is defined using $\mathrm{ev}_1 \colon (EY)^{\Delta[1]} \to EY$ and $p$ is defined using $\mathrm{ev}_0 \colon (EY)^{\Delta[1]} \to EY$. The map $p$ is a fibration: it can be factored

$$(EY)^{\Delta[1]} \times_{EY} EX \to (EY)^{\partial\Delta[1]} \times_{EY} EX \approx EY \times EX \to EY,$$

where both maps are fibrations since $EX$ and $EY$ are fibrant. By hypothesis, the dotted arrow exists. Furthermore, $\pi$ is a trivial fibration, and hence $i$ and $j$ are weak equivalences, and we can conclude that $f$ is a weak equivalence. $\square$

7.7. **A useful lemma.** It is convenient to give here the following generalization of (7.6), which is used in §8.

**Lemma 7.8.** *Given the hypotheses of (7.6), suppose that $F \colon T\text{-alg} \to s\mathcal{S}^{\mathcal{J}}$ is a degreewise functor which commutes with filtered colimits and reflexive coequalizers in each degree. Then $F(f)$ is a weak equivalence in $s\mathcal{S}^{\mathcal{J}}$.*

*Proof.* Consider the diagram

$$\begin{tikzcd}
FX \arrow[r] \arrow[d, "Ff"] & F\left((EY)^{\Delta[1]} \times_{EY} EX\right) \arrow[r] \arrow[d] & (EFY)^{\Delta[1]} \times_{EFY} EFX \arrow[d] \\
FY \arrow[r] \arrow[ur] & FEY \arrow[r] & EFY.
\end{tikzcd}$$

The left-hand side is obtained by applying $F$ to the square used in the proof of (7.6). By (4.12) the functor $F$ must be representable by some right $T$-module, and therefore the horizontal maps on the right-hand side are obtained using (5.1) and (5.2), and the right-hand square commutes. The top and bottom rows of the rectangle are weak equivalences by the same arguments as used in the proof of (7.6), and hence we conclude that $Ff$ is a weak equivalence. $\square$

## 8. Homotopy invariance properties

This section is dedicated to giving criteria for functors to preserve weak equivalences. As a corollary (8.6) of these results, we will see that the homotopy theory of $T$-algebras depends only on the weak homotopy type of the simplicial theory $T$.

**Theorem 8.1.** *Let $T$ be an $\mathcal{I}$-sorted theory, and $f \colon X \to Y$ a weak equivalence between cofibrant $T$-algebras, and let $F \colon T\text{-alg} \to s\mathcal{S}^{\mathcal{J}}$ be a degreewise functor which commutes with filtered colimits and reflexive coequalizers (i.e., a right $T$-module). Then $F(f)$ is a weak equivalence.*

*Proof.* If $f$ is a trivial cofibration, this is (7.8). The theorem follows using (2.6). $\square$

**Proposition 8.2.** *Let $A \to B \in s\mathcal{S}^{f\mathcal{S}(\mathcal{I},\mathcal{J})}$ be a weak equivalence, and let $X \in s\mathcal{S}^{\mathcal{I}}$. Then the induced map $A(X) \to B(X)$ is a weak equivalence in $s\mathcal{S}^{\mathcal{J}}$.*

*Proof.* We can first reduce to the case when $X$ is a *discrete* graded simplicial set, using the diagonal principle (1.1) and the fact that $A(X)$ (and similarly $B(X)$) can be obtained as the diagonal of the simplicial object in $s\mathcal{S}^{\mathcal{J}}$ given by $[n] \mapsto A(X_n)$, where $X_n$ is the $n$th simplicial degree of $X$. Next note

that it is enough to show that the conclusion holds when $X$ is both discrete and *finite*, since every graded set is a filtered colimit of its finite subsets, and $A(-)$ and $B(-)$ commute with such colimits. Now we are done, since $A \to B$ is a weak equivalence exactly when $A(K) \to B(K) \in s\mathcal{S}^{\mathcal{J}}$ is one for all $K \in f\mathcal{S}/\mathcal{J}$. ☐

**Theorem 8.3.** *Let $f\colon M \to M'$ be a map of right $T$-modules. The following are equivalent.*

(1) *The map $f$ is a weak equivalence of right $T$-modules.*
(2) *For every $T$-algebra $X$ of the form $X = T(K)$ with $K \in f\mathcal{S}/\mathcal{J} \subset s\mathcal{S}^{\mathcal{J}}$, the induced map $M \circ_T X \to M' \circ_T X$ is a weak equivalence.*
(3) *For every* cofibrant *$T$-algebra $X$, the induced map $M \circ_T X \to M' \circ_T X$ is a weak equivalence.*

*Proof.* The equivalence of (1) and (2) is immediate, since the $j$th graded piece of $M \circ_T T(K)$ is $M(j, K)$. Since for any $K \in f\mathcal{S}/\mathcal{J} \subset s\mathcal{S}^{\mathcal{J}}$, $X = T(K)$ is a cofibrant $T$-algebra, (3) implies (2).

To show that (1) implies (3), let $Y$ be a simplicial object in $T$-alg defined by $[n] \mapsto Y_{n,*} = T^{n+1}X$; then $\operatorname{diag}(Y) \in T$-alg is $s$-free by (6.4), and hence is cofibrant, and $\operatorname{diag}(Y) \to X$ is a weak equivalence by the existence of a contracting homotopy. Now consider

$$\begin{tikzcd}
\operatorname{diag}(M \circ_T Y) \arrow[r, equal] \arrow[d, "g"'] & M \circ_T (\operatorname{diag} Y) \arrow[r, "\sim"] \arrow[d] & M \circ_T X \arrow[d, "f \circ_T X"] \\
\operatorname{diag}(M' \circ_T Y) \arrow[r, equal] & M' \circ_T (\operatorname{diag} Y) \arrow[r, "\sim"] & M' \circ_T X.
\end{tikzcd}$$

The maps marked $\sim$ are weak equivalences by (8.1), so to show that $f \circ_T X$ is a weak equivalence it suffices to show that $g$ is. By the diagonal principle (1.1), it suffices to show that $M \circ_T T^{n+1}X \approx M \circ T^n X \to M' \circ_T T^{n+1}X \approx M' \circ T^n X$ is a weak equivalence for $n \geq 0$; this is (8.2). ☐

**Corollary 8.4.** *Let $f\colon X \to X'$ be any weak equivalence of $T$-algebras. Then for any* cofibrant *right $T$-module $M$, the induced map $M \circ_T X \to M \circ_T X'$ is a weak equivalence.*

*Proof.* The functors $- \circ_T X, - \circ_T X'\colon I, T\text{-mod} \to s\mathcal{S}^{\mathcal{J}}$ are represented by an appropriate bimodules $N_X$ and $N_{X'}$, as described in (4.6). We claim that the map $N_X \to N_{X'}$ induced by $f$ is a weak equivalence, which means that we can derive the corollary as a special case of (8.3). To see that $N_X \to N_{X'}$ is a weak equivalence, it suffices to show that it induces a weak equivalence when applied to a free "algebra", by (8.3). Translated, this means that we must show that $M \circ_T X \to M \circ_T X'$ is a weak equivalence when $M$ is a free right $T$-module. In fact, this is the case whenever $M \approx A \circ T$ for some $A \in s\mathcal{S}^{f\mathcal{S}(\mathcal{J},*)}$, by (8.2), and so is in particular true for free objects. ☐

*Remark* 8.5. If $M$ is a $T, S$-bimodule, then (8.1) implies that the induced functor $M \circ_S -\colon S\text{-alg} \to T\text{-alg}$ preserves all weak equivalences between cofibrant $S$-algebras. Therefore, there is an induced *left derived functor* $M \circ_S^{\mathbf{L}} -\colon \operatorname{Ho} S\text{-alg} \to \operatorname{Ho} T\text{-alg}$. Similar considerations show that if $X$ is an $S$-algebra, then the induced functor $- \circ_S X\colon T, S\text{-mod} \to T\text{-alg}$ preserves all weak equivalence between all bimodules which are cofibrant as right $S$-modules, and hence induces a left derived functor $- \circ_S^{\mathbf{L}} X\colon \operatorname{Ho} T, S\text{-mod} \to \operatorname{Ho} T\text{-alg}$.

Furthermore, (8.3) and (8.4) show that the two ways of defining $M \circ_S^{\mathbf{L}} X$ are isomorphic in $\operatorname{Ho} T$-alg; that is, there is a well-defined *derived pairing* $- \circ_S^{\mathbf{L}} -\colon \operatorname{Ho} T, S\text{-mod} \times \operatorname{Ho} S\text{-alg} \to \operatorname{Ho} T\text{-alg}$.

**Corollary 8.6.** *Let $\varphi\colon S \to T$ be a morphism of simplicial $\mathcal{J}$-sorted theories. Then the induced Quillen adjoint pair* (7.3)

$$\varphi_*\colon S\text{-alg} \rightleftarrows T\text{-alg} : \varphi^*$$

*is a Quillen equivalence if and only if $\varphi$ is a weak equivalence of theories.*

*Proof.* First, note that the pair is a Quillen equivalence if and only if the adjunction map $X \to \varphi^*\varphi_* X$ is a weak equivalence for every cofibrant $S$-algebra $X$. This is because, given $f\colon \varphi_* X \to Y \in T\text{-alg}$, the adjoint map factors

$$X \to \varphi^*\varphi_* X \xrightarrow{\varphi^* f} \varphi^* Y,$$

and $\varphi^* f$ is a weak equivalence if and only if $f$ is. The result now follows from (8.3), since the adjunction map is isomorphic to $S \circ_S X \to T \circ_S X$. □

## 9. A criterion for properness

In this section we give a criterion for a category of simplicial algebras over a theory to be left proper. The proof is adapted with some changes from an argument of Dwyer and Kan [DK80, §8], who use it to show that simplicially enriched categories with a fixed object set form a proper model category.

**Theorem 9.1.** *Let $T$ be an $\mathcal{I}$-sorted simplicial theory. The following are equivalent.*

1. *The model category $T$-alg is proper.*
2. *For each finite $\mathcal{I}$-graded set $K \in f\mathcal{S}/\mathcal{I} \subset \mathcal{S}^{\mathcal{I}} \subset s\mathcal{S}^{\mathcal{I}}$, the functor $T\text{-alg} \to T\text{-alg}$ given by $X \mapsto X \amalg^T T(K)$ carries weak equivalences to weak equivalences.*

*Remark* 9.2. Note that it suffices in condition (2) of (9.1) to take only those $K$ whose underlying set is singleton. In particular, if $\mathcal{I}$ is singleton, then the theorem says that $T$-alg is proper if and only if the functor $X \mapsto X \amalg^T T(1)$ preserves weak equivalences.

*Proof.* We have already seen that $T$-alg is always right proper (7.2), so we need only consider left properness. That (1) implies (2) follows by observing that if $f\colon X \to Y \in T\text{-alg}$, then the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X \amalg^T T(K) \arrow[d, "g"] \\ Y \arrow[r] & Y \amalg^T T(K) \end{tikzcd}$$

is a pushout square in $T$-algebras in which the top arrow is a cofibration; properness implies that $g$ is a weak equivalence if $f$ is.

To show (2) implies (1), we must show that for any cofibration $i\colon U \to V$,

(*) the functor $-\amalg^T_U V : U\backslash T\text{-alg} \to V\backslash T\text{-alg}$ carries weak equivalences to weak equivalences.

We proceed by a series of reductions. First, it suffices to show (*) when $i$ is an $s$-free map, since cofibrations are strong retracts of such.

Next, it suffices to show (*) for $i$ of the form $T(j)\colon T(K) \to T(L)$ where $j\colon K \to L$ is an inclusion of $\mathcal{I}$-graded simplicial sets. This is because any $s$-free map can be written as a directed colimit of a series of maps, each of which is a pushout along a map of the form $T(j)$, and because weak equivalences are preserved by directed colimits.

Define $\mathcal{B}(X, U, V)$ to be the simplicial object in $T$-alg given by

$$[n] \mapsto \mathcal{B}_n(X, U, V) = X \amalg^T \left(\coprod_n^T U\right) \amalg^T V.$$

We claim that if $i = T(j)\colon T(K) \to T(L)$, then the evident augmentation $\operatorname{diag}\mathcal{B}(X, U, V) \to X \amalg^T_U V$ is a weak equivalence. In fact, in each *internal* degree $m$ we have that $L_m \approx K_m \amalg K'_m$ for some $\mathcal{I}$-graded set $K'_m$, and thus

$$[n] \mapsto \mathcal{B}_n(X_m, U_m, V_m) \approx X_m \overset{T_m}{\amalg} \coprod_n^{T_m} U_m \overset{T_m}{\amalg} V_m \approx X_m \overset{T_m}{\amalg} T_m(\coprod_n K_m \amalg K_m \amalg K'_m),$$

which augments to $X_m \amalg^T_{T(K_m)} T(L_m) \approx X_m \amalg^{T_m} T_m(K'_m)$. There is an evident contracting homotopy using the inclusion $K'_m \to K_m \amalg K'_m$, showing that $\mathcal{B}(X_m, U_m, V_m) \to (X \amalg^T_U V)_m$ is a weak equivalence of (graded) simplicial sets, and hence the claim follows using the diagonal principle (1.1).

Next, it suffices to show (*) for $i$ of the form $T(0) \to T(K)$ for $K \in s\mathcal{S}^{\mathcal{I}}$; that is, to show that the functor $X \mapsto X \amalg^T T(K)$ preserves weak equivalences. This follows using the diagonal principle and the above claim, since then for $n \geq 0$ each $X \mapsto X \amalg^T T(\coprod_n K \amalg L)$ must preserve weak equivalences.

Next, it suffices to show (*) for $i$ of the form $T(0) \to T(K)$ where $K$ is a discrete graded simplicial set; this follows by another application of the diagonal principle to $[n] \mapsto X \amalg^T T(K_n)$, the diagonal of which is $X \amalg^T T(K)$.

The theorem now follows using the fact that $X \amalg^T T(K)$, with $K$ discrete, is a filtered colimit over the diagram of all finite subobjects of $K$, and that weak equivalences are preserved by filtered colimits. □

## 10. Free theories and trees

In this section we give the explicit construction of a free theory over graded sets, and use this to derive some results needed for the proof of (11.1). Essentially, we show (10.8) that a coproduct of two *free* theories is free as a right module over one of these theories. That free theories may be described in terms of trees is an observation of Boardman [Boa71], [BV73]. The point of view we take here is that free theories are essentially the same as free *operads* (more precisely, free $\Sigma$-operads, i.e., ones in which symmetric groups do not act), which can also be described using trees. Our definitions of trees are based on those of [GJ], and on ones given in an early version of [GHb].

10.1. **Trees.** A **totally ordered tree** $\mathcal{T}$ (or simply **tree**) is an oriented contractible graph which

(1) has a (possibly empty) finite set of vertices, such that
(2) each vertex has a (possibly empty) finite totally ordered set of input edges,
(3) each vertex has exactly one output edge, and
(4) there is exactly one edge of $\mathcal{T}$ which is not the output edge of a vertex.

Let $\mathrm{in}(v)$ denote the ordered set of input edges of a vertex $v$, and let $\mathrm{out}(v)$ denote the unique output edge. The external edges of a tree $\mathcal{T}$ consist of a unique output edge $\mathrm{out}(\mathcal{T})$ and a set of input edges $\mathrm{in}(\mathcal{T})$, which acquires a total ordering in an evident way from the orderings of the $\mathrm{out}(v)$. The output edge of a tree is not an input edge, *except* for the case of a tree which has an empty set of vertices; this is called the **trivial tree**, and it has a unique edge.

We fix a total ordering of each finite set $n \in f\mathcal{S}$, so that there is a unique order preserving bijection between $\mathrm{in}(v)$ (resp. $\mathrm{in}(\mathcal{T})$) and some $n$, making it convenient to identify these sets when necessary.

There is an evident notion of isomorphism of trees, and we will identify isomorphic trees.

Let $\mathcal{I}$ be a set. An **$\mathcal{I}$-tree** is a tree $\mathcal{T}$ together with a choice of an element $i(e) \in \mathcal{I}$ for each edge $e$ of $\mathcal{T}$; in other words, the set of edges of $\mathcal{T}$ is an $\mathcal{I}$-graded set. To each vertex of an $\mathcal{I}$-tree one can associate an element $i(v) \in \mathbb{N}(\mathcal{I}) \approx \mathrm{ob}(\mathcal{I} \times f\mathcal{S}/\mathcal{I})$, namely the pair $(i(\mathrm{out}(v)), i|_{\mathrm{in}(v)} \colon \mathrm{in}(v) \to \mathcal{I})$. Similarly, to an $\mathcal{I}$-tree there is an associated element $i(\mathcal{T}) \in \mathbb{N}(\mathcal{I})$, namely the pair $(i(\mathrm{out}(\mathcal{T})), i \colon \mathrm{in}(\mathcal{T}) \to \mathcal{I})$.

Let $A \in \mathcal{S}^{\mathbb{N}(\mathcal{I})}$. An **$A$-labelled $\mathcal{I}$-tree** is a tree $\mathcal{T}$ together with a choice, for each vertex $v$ of $\mathcal{T}$, of an element $a(v) \in A(i(v))$. The set of isomorphism classes of $A$-labelled trees is naturally a $\mathbb{N}(\mathcal{I})$-graded set, denoted $\mathcal{Q}A$, with the $K \in \mathbb{N}(\mathcal{I})$ graded piece isomorphic to

$$(\mathcal{Q}A)(K) \approx \coprod_{\substack{\text{trees } \mathcal{T}, \\ i(\mathcal{T})=K}} \prod_{\substack{\text{vertices} \\ v \text{ of } \mathcal{T}}} A(i(v)).$$

If $\mathcal{T}$ is an $A$-labelled $\mathcal{I}$-tree with input edges $\mathrm{in}(\mathcal{T})$, and if for each $k \in \mathrm{in}(\mathcal{T})$ the $\mathcal{T}_1, \dots, \mathcal{T}_n$ are $A$-labelled $\mathcal{I}$-trees such that $i(\mathrm{out}(\mathcal{T}_k)) = i(k)$, then we can form a tree $\mathcal{T}[\mathcal{T}_1, \dots, \mathcal{T}_n]$ by **grafting** $\mathcal{T}_k$ at the edge $k$, obtaining a new $A$-labelled $\mathcal{I}$-tree.

10.2. **Description of free theories by trees.** Suppose $\mathcal{F}\colon \mathcal{S}^{\mathbb{N}(\mathcal{I})} \to \mathcal{T}(\mathcal{I})$ (the **free theory** functor) and $\mathcal{S}\colon \mathcal{S}^{\mathbb{N}(\mathcal{I})} \to \mathcal{S}^{f\mathcal{S}(\mathcal{I})}$ (the **free series** functor, as in (3.9)) denote the left adjoints to the corresponding forgetful functors.

For $A, B \in \mathcal{S}^{\mathbb{N}(\mathcal{I})}$, define $A * B \in \mathcal{S}^{\mathbb{N}(\mathcal{I})}$ by

$$(A * B)(i, f\colon n \to \mathcal{I}) = \coprod_{g\colon m \to \mathcal{I} \in f\mathcal{S}/\mathcal{I}} A(i, g) \times \left( \coprod_{\substack{h\colon n \to m \\ \text{weak monot.}}} \prod_{k \in m} B(g(k), f|h^{-1}(k)) \right),$$

where the second coproduct is taken over the set of *weakly monotone* maps $h\colon n \to m$ in $f\mathcal{S}$ (i.e., $i \leq j$ implies $h(i) \leq h(j)$), and $h^{-1}(k) \subset n$ is identified bijectively with an object of $f\mathcal{S}$ via the ordering induced as a subset of $n$. Let $\delta \in \mathcal{S}^{\mathbb{N}(\mathcal{I})}$ denote the object with

$$\delta(i, f\colon n \to \mathcal{I}) = \begin{cases} * & \text{if } n = 1 \text{ and } f(1) = i, \\ \varnothing & \text{otherwise.} \end{cases}$$

(If $\mathcal{I}$ is singleton, these become

$$(A * B)(n) = \coprod_m A(m) \times \coprod_{i_1 + \cdots + i_m = n} B(i_1) \times \cdots \times B(i_m), \qquad \delta(n) = \begin{cases} * & \text{if } n = 1, \\ \varnothing & \text{otherwise.} \end{cases} .)$$

**Lemma 10.3.** *The category $\mathcal{S}^{\mathbb{N}(\mathcal{I})}$ admits the structure of a monoidal category, with the monoidal product given by $*$ and with unit object $\delta$. Furthermore, the functor $\mathcal{S}\colon \mathcal{S}^{\mathbb{N}(\mathcal{I})} \to \mathcal{S}^{f\mathcal{S}(\mathcal{I})}$ admits the structure of a monoidal functor for which $I \approx \mathcal{S}\delta$ and $\mathcal{S}(A * B) \approx \mathcal{S}A \circ \mathcal{S}B$.*

*Proof.* Recall from (3.8) that $\mathcal{S}^{f\mathcal{S}(\mathcal{I})}$ is equivalent to a full subcategory of the category of endofunctors on $\mathcal{S}^{\mathcal{I}}$. There is an evident explicit isomorphism $\mathcal{S}A(\mathcal{S}B(X)) \approx \mathcal{S}(A * B)(X)$ natural in $X \in \mathcal{S}^{\mathcal{I}}$, as can be seen by applying (3.9). More explicit computations show that the monoidal structure on $\mathcal{S}^{f\mathcal{S}(\mathcal{I})}$ restricts to $\mathcal{S}^{\mathbb{N}(\mathcal{I})}$ along $\mathcal{S}\colon \mathcal{S}^{\mathbb{N}(\mathcal{I})} \to \mathcal{S}^{f\mathcal{S}(\mathcal{I})}$. □

**Proposition 10.4.** *$\mathcal{F}A \approx \mathcal{S}(\mathcal{Q}A)$ as objects of $\mathcal{S}^{f\mathcal{S}(\mathcal{I})}$.*

*Remark* 10.5. The object $\mathcal{Q}A$ is nothing more than the free $\Sigma$-operad on $A$ (cf. [GJ]). Thus this proposition relates the free $\Sigma$-operad on $A$ with the free theory on $A$.

*Proof of Proposition 10.4.* It is enough to show that $\mathcal{Q}A$ is the *free monoid* with respect to the $*$-product on $\mathcal{S}^{\mathbb{N}(\mathcal{I})}$; that is, maps $A \to M \in \mathcal{S}^{\mathbb{N}(\mathcal{I})}$ are in bijective correspondence with maps $\mathcal{Q}A \to M$ of monoids. Then from (10.3) it follows formally that $\mathcal{S}(\mathcal{Q}A)$ is the free monoid with respect to the $\circ$-product, i.e., it is a free theory.

To make $\mathcal{Q}A$ into a monoid with respect to the $*$ structrue, let $\delta \to \mathcal{Q}A$ be the map classifying the trivial trees, and let $\mathcal{Q}A * \mathcal{Q}A \to \mathcal{Q}A$ be the evident map describing grafting of trees. Now note that $\mathcal{Q}A$ is precisely the formula for the free $\Sigma$-operad on $A$. □

10.6. **Essentially labelled trees.** If $\mathcal{T}$ is a tree, we say that $\mathcal{T}' \subset \mathcal{T}$ is a **rooted subtree** if it is a subtree such that $\mathrm{out}(\mathcal{T}') = \mathrm{out}(\mathcal{T})$. Given any rooted subtree $\mathcal{T}'$ of $\mathcal{T}$ there is a unique way to write $\mathcal{T}$ as a graft $\mathcal{T}'[\mathcal{T}_1, \dots, \mathcal{T}_n]$ for some subtrees $\mathcal{T}_1, \dots, \mathcal{T}_n$.

Let $\mathcal{T} \in \mathcal{Q}(A \amalg B)$. Let $e_B(\mathcal{T})$ denote the minimal rooted subtree of $\mathcal{T}$ which contains all of the vertices which are labelled by $B$; if no vertices are labelled by $B$ then $e_B(\mathcal{T})$ is a trivial tree. Say that $\mathcal{T} \in \mathcal{Q}(A \amalg B)$ is $B$**-essential** if $e_B(\mathcal{T}) = \mathcal{T}$, and write $\mathcal{Q}_e(A, B) \subset \mathcal{Q}(A \amalg B)$ for the sub-$\mathbb{N}(\mathcal{I})$-graded set of $B$-essential trees. We thus have shown

**Lemma 10.7.** *Every $\mathcal{T} \in \mathcal{Q}(A \amalg B)$ can be written uniquely as the grafting of a $B$-essential $\mathcal{I}$-tree $\mathcal{T}'$ with $\mathcal{I}$-trees $\mathcal{T}_1, \dots, \mathcal{T}_n$ labelled only by $A$.*

**Proposition 10.8.** *$\mathcal{F}(A \amalg B) \approx \mathcal{S}(\mathcal{Q}_e(A, B)) \circ \mathcal{F}A$ as objects in the category of right $\mathcal{F}A$-modules.*

*Proof.* Using (10.3) and (10.4), this amounts to showing that $\mathcal{Q}(A \amalg B) \approx \mathcal{Q}_e(A, B) * \mathcal{Q}A$, which is a direct translation of (10.7). ☐

**Proposition 10.9.** *The diagram $\mathcal{Q}_e(A, \varnothing) \to \mathcal{Q}_e(A, B) \rightrightarrows \mathcal{Q}_e(A, B \amalg B)$ is an equalizer of $\mathbb{N}(\mathcal{I})$-graded sets, where the parallel maps are those induced by the two inclusions of $B$ into $B \amalg B$.*

*Proof.* If $\mathcal{T} \in \mathcal{Q}_e(A, B)$ has the same image under the two maps, then it can have no vertices labelled by $B$, and hence must be a trivial tree. There is exactly one trivial tree for each element of $\mathcal{I}$, and $\mathcal{Q}_e(A, \varnothing)$ contains only these. ☐

## 11. Cofibrations of theories and properness

In this section we show (11.4) that cofibrant theories give rise to proper model categories.

**Theorem 11.1.** *Let $\varphi\colon T \to U$ be a cofibration between cofibrant simplicial theories. Then $U$ is cofibrant as a right $T$-module.*

Taking (11.1) together with (8.4) immediately gives

**Corollary 11.2.** *If $\varphi\colon T \to U$ is a cofibration between cofibrant simplical theories, then $\varphi_*\colon T\text{-alg} \to U\text{-alg}$ preserves all weak equivalences.*

*Proof of* (11.1). We first show that it suffices to assume that $T$ is an $s$-free theory and that $\varphi$ is an $s$-free map of theories. In fact, using the model category structure we see that $\varphi$ is a retract of a map $\varphi'\colon T' \to U'$, where $T'$ and $\varphi'$ are $s$-free. Then there are maps $U \to U' \circ_{T'} T \to U$ of right $T$-modules, and the composite of these maps is the identity, making $U$ a retract of $U' \circ_{T'} T$ as a right $T$-module. If $U'$ is cofibrant as a right $T'$-module, then $U' \circ_{T'} T$ is cofibrant as a right $T$-module (since the functor $- \circ_{T'} T\colon I, T'\text{-mod} \to I, T\text{-mod}$ is the left adjoint of a Quillen pair), and hence $U$ is too.

Now suppose that $T$ and $\varphi$ are $s$-free. Thus $T_n \approx \mathcal{F}A_n$ and $U_n \approx \mathcal{F}(A_n \amalg B_n)$, where $A_n$ and $B_n$ are free degeneracy diagrams in $s\mathcal{S}^{\mathbb{N}(\mathcal{I})}$. Then by (10.8) we have $U_n \approx \mathcal{S}(\mathcal{Q}_e(A_n, B_n)) \circ T_n$. Thus, it suffices to show that $[n] \mapsto \mathcal{Q}_e(A_n, B_n)$ is a free degeneracy diagram in $\mathcal{S}^{\mathbb{N}(\mathcal{I})}$. Now $\mathcal{Q}_e(A, B) \subset \mathcal{Q}(A \amalg B)$, and $\mathcal{Q}(A \amalg B)$ is free by the hypotheses that $T$ and $\varphi$ be $s$-free. By (6.2) it suffices to show that $\mathcal{Q}_e(A, B)$ is closed inside of $\mathcal{Q}(A \amalg B)$. That is, if $\mathcal{T} \in \mathcal{Q}(A \amalg B)$ and $\sigma \in \Delta_+$ such that $\mathcal{T}\sigma \in \mathcal{Q}_e(A, B)$, then $\mathcal{T} \in \mathcal{Q}_e(A, B)$. The operator $\sigma$ acts on $\mathcal{T}$ by relabeling the vertices of $\mathcal{T}$ according to the way $\sigma$ acts on $A$ and $B$ separately, and it does not change the underlying shape of the tree or whether a given vertex is labelled by $A$ or $B$; hence, if $\sigma(\mathcal{T})$ is $B$-essential, then so is $\mathcal{T}$. ☐

Given $K \in \mathcal{S}^{\mathcal{I}}$, let $\epsilon K \in \mathcal{S}^{\mathbb{N}(\mathcal{I})}$ denote the object with $(\epsilon K)(\alpha, 0 \to \mathcal{I}) = K_\alpha$, and $(\epsilon K)(\alpha, n \to \mathcal{I}) = \varnothing$ for $n > 0$.

**Lemma 11.3.** *The theory $T_{T(K)}$ (4.8) is isomorphic to $T \amalg^{\mathcal{F}} \mathcal{F}(\epsilon K)$, where $\mathcal{F}(\epsilon K)$ is the free $\mathcal{I}$-sorted theory on $\epsilon K$, and the coproduct is taken in the category of $\mathcal{I}$-sorted theories.*

*Proof.* Using the endomorphism theory technology of (4.8), it is easy to see that $\mathcal{F}(\epsilon K)\text{-alg} \approx K \backslash s\mathcal{S}^{\mathcal{I}}$. By (4.10) we see that algebras over $T \coprod^{\mathcal{F}} \mathcal{F}(\epsilon K)$ are the same as $T$-algebras $X$ equipped with a map $K \to X$ of graded sets, or equivalently, the same as $T$-algebras $X$ equipped with a map $T(K) \to X$ of $T$-algebras. ☐

**Corollary 11.4.** *If $T$ is a cofibrant simplicial theory, then $T\text{-alg}$ is a proper model category.*

*Proof.* Suppose that $K \in f\mathcal{S}/\mathcal{I} \subset s\mathcal{S}^{\mathcal{I}}$. By (11.3), $T \to T_{T(K)}$ is a cofibration between cofibrant theories, and thus $T_{T(K)} \circ_T -\colon T\text{-alg} \to T_{T(K)}\text{-alg} \approx T(K)\backslash T\text{-alg}$ carries weak equivalences to weak equivalences by (11.2). Since there is an isomorphism $T_{T(K)}(X) \approx X \amalg^T T(K)$ of underlying $T$-algebras, it follows that $T\text{-alg}$ is proper by (9.1). ☐

## 12. Proofs of the theorems

*Proof of Theorems A and B.* Given a simplicial theory $T$, one can construct a weak equivalence $S \to T$ from a cofibrant theory $S$, since simplicial theories are a model category (7.5). Then $S$-alg is a proper simplicial model category by (11.4), and the induced Quillen pair $S\text{-alg} \rightleftarrows T\text{-alg}$ is a Quillen equivalence by (8.6). □

*Proof of Theorem C.* Recall that $T$-alg being pointed means that the initial object $T(0)$ is isomorphic to the terminal object, denoted $*$. Choose $\varphi\colon S \to T$ as in the proof of Theorem B, so that $S$-alg is proper and is Quillen equivalent to $T$-alg via $\varphi$. The initial object in $S$-alg is $S(0)$, which is not in general the terminal object. But since $S \to T$ is a weak equivalence, $S(0)$ is weakly equivalent to $T(0) \approx *$.

Let $S_*$ denote the theory of $S$-algebras under $*$ as in (4.8), so that $S_*\text{-alg} \approx *\backslash S\text{-alg}$. We have restriction functors $T\text{-alg} \to S_*\text{-alg} \to S\text{-alg}$ factoring $\varphi^*$ and hence maps

$$S \xrightarrow{\psi'} S_* \xrightarrow{\psi''} T$$

of theories factoring the weak equivalence $\varphi$. Since $S$-alg is proper, the Quillen pair induced by $\psi'$ is a Quillen equivalence by (2.7), and hence a weak equivalence by (8.6). Hence $\psi''$ is a weak equivalence and so induces a Quillen equivalence between $S_*$-alg and $T$-alg. The theorem is now proved, since $S_*$-alg is a pointed category, and is proper by (2.8 (ii)). □

An effective monomorphism $X \to Y$ in a category with pushouts is a map such that $X \to Y \rightrightarrows Y \amalg_X Y$ is an equalizer.

**Lemma 12.1.** *If $T$ is a cofibrant simplicial theory, then cofibrations in $T$-alg are effective monomorphisms.*

*Proof.* We first show that it suffices to assume that $T$ is $s$-free. In general, $T$ is a retract of some $s$-free $T'$. Let $i\colon X \to Y$ be a cofibration of $T$-algebras. Write $X' = T' \circ_T X$ and $Y' = T' \circ_T Y$. Then the diagram $X \to Y \rightrightarrows Y \amalg_X^T Y$ is a retract of the diagram obtained by applying $T' \circ_T -$ to it, which is $X' \to Y' \rightrightarrows Y' \amalg_{X'}^{T'} Y'$, and the map $i' = T' \circ_T i\colon X' \to Y'$ is a cofibration of $T'$-algebras. If we know that $i'$ is an effective monomorphism, then this diagram is an equalizer, and so is any retract of it, whence $i$ is an effective monomorphism.

Now assume $T$ is $s$-free. We can also assume that $i$ is an $s$-free map, since retracts of effective monomorphisms are again effective monomorphisms. To show that $i$ is an effective mono, it suffices to check it in each simplicial degree. Thus, we must show that for $A \in \mathcal{S}^{\mathbb{N}(\mathcal{I})}$, $X \in \mathcal{F}A$-alg, and $K \in \mathcal{S}^{\mathcal{I}}$, the diagram

$$X \to X \amalg^{\mathcal{F}A} (\mathcal{F}A)(K) \rightrightarrows X \amalg^{\mathcal{F}A} (\mathcal{F}A)(K \amalg K)$$

is an equalizer. Using (11.3) and (10.8) this is the same as

$$\mathcal{S}(\mathcal{Q}_e(A, \varnothing)) \circ X \to \mathcal{S}(\mathcal{Q}_e(A, \epsilon K)) \circ X \rightrightarrows \mathcal{S}(\mathcal{Q}_e(A, \epsilon K \amalg \epsilon K)) \circ X,$$

where $\epsilon K$ is as defined in §11, and the lemma now follows easily using (10.9). □

We note that the conclusion of (12.1) does not hold for a general theory. For a counterexample, take $\mathcal{I}$ singleton, and let $T$ be the unique theory with $T(0) = \varnothing$ and $T(n) = *$ for $n > 0$. The category of $T$-algebras has exactly two objects: $\varnothing$ and $*$. The unique map $\varnothing \to *$ is a monomorphism, but is not effective!

*Proof of Theorem D.* A model category $\mathbf{M}$ is **cellular** in the sense of Hirschhorn [Hir] if it is a cofibrantly generated model category with sets $I$ and $J$ of generating cofibrations and trivial cofibrations with the property that

(1) the domains and codomains of the elements of $I$ are "compact",

(2) the domains of the elements of $J$ are "small relative to $I$", and
(3) the cofibrations are effective monomorphisms.

Axioms (1) and (2) say that mapping out of the domains and codomains of the generators commutes with certain kinds of directed colimits (for the precise notions, refer to [Hir]). They certainly hold for categories of algebras over a simplical theory, since in that case the domains and codomains of the generators are "small" in the sense that mapping out of them commutes with arbitrary filtered colimits. Axiom (3) holds for a cofibrant theory by (12.1), giving the result for the hypotheses of Theorem B. If Axiom (3) holds in a model category, it also holds in all undercategories, and this gives the result for the hypotheses of Theorem C. $\square$

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